A FAMILY OF NEW-WAY INTEGRALS FOR THE STANDARD $\mathcal{L}$-FUNCTION OF CUSPIDAL REPRESENTATIONS OF THE EXCEPTIONAL GROUP OF TYPE G_2

AVNER SEGAL

ABSTRACT. Let $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})$ be a standard twisted partial $\mathcal{L}$-function of degree 7 of the cuspidal automorphic representation π of the exceptional group of type G_2. In this paper we construct a family of Rankin-Selberg integrals representing this $\mathcal{L}$-function. As an application, we prove that the representations attaining certain prescribed poles are exactly the representations attained by θ-lift from a group of finite type.

Contents

1. Introduction

$\mathcal{L}$-functions are one of the central objects of modern number theory, representation theory, the theory of automorphic forms and other areas of mathematics. They are used in order to achieve algebraic and arithmetic information by means of analytic inquiries. The use of $\mathcal{L}$-functions in number theory goes back to Dirichlet who used them in 1837 [5] to prove that any arithmetic progression, with coprime coefficients, contains infinitely many prime numbers.

Let G be a reductive group defined over a number field F. Given a non-Archimedean place ν of a number field F and an unramified representation π_ν of $G(F_\nu)$, there is a semisimple conjugacy class $t_{\pi_\nu} \in {}^L G$, called the Satake parameter of π_ν. For a finite dimensional representation ρ of ${}^L G$ the local $\mathcal{L}$-factor is defined by

$$\mathcal{L}(s, \pi_\nu, \rho) = \frac{1}{\det(\mathbb{1} - \rho(t_{\pi_\nu}) q_\nu^{-s})},$$

where q_ν is the cardinality of the residue field of F_ν.

Given an irreducible automorphic representation $\pi = \otimes_\nu \pi_\nu$ of $G\left(\mathbb{A}_F\right)$ and a finite set of places S, such that π_ν is unramified for $\nu \notin S$, the global partial $\mathcal{L}$-function is defined by

$$\mathcal{L}^S\left(s, \pi, \rho\right) = \prod_{\nu \notin S} \mathcal{L}\left(s, \pi_\nu, \rho\right) \ .$$

Langlands conjectured that this product converges for $Re(s) >> 0$ and admits a meromorphic continuation to the whole complex plane. The most effective way known to prove this conjecture, for various cases, is by attaching to $\mathcal{L}^S\left(s, \pi, \rho\right)$ an integral representation with convenient analytic properties. One method of doing this is the Rankin-Selberg method. For a comprehensive survey of the Rankin-Selberg method consult [4].

Let π be an irreducible cuspidal representation of the exceptional group $G_2\left(\mathbb{A}_F\right)$ and let χ be a Hecke character of $GL_1\left(\mathbb{A}_F\right)$. The Langlands dual group of G_2 is isomorphic to $G_2\left(\mathbb{C}\right)$ and the Langlands dual group of GL_1 is isomorphic to $\mathbb{C}^\times$. We denote by $\mathfrak{st}$ the irreducible seven-dimensional complex representation of $G_2\left(\mathbb{C}\right) \times \mathbb{C}^\times$. Our **main objective** is to attach an integral representation for the partial $\mathcal{L}$-function $\mathcal{L}\left(s, \pi, \chi, \mathfrak{st}\right) = \mathcal{L}\left(s, \pi \boxtimes \chi, \mathfrak{st}\right)$ and to prove its meromorphic continuation.

In this paper we consider a family of zeta integrals $\mathcal{Z}_E\left(\chi, s, \varphi, f\right)$ parametrized by étale cubic algebras E over F. Étale cubic algebras over F also parametrize the non-degenerate Fourier coefficients along the Heisenberg unipotent subgroup U of G_2. For any such algebra E we denote by Ψ_E the corresponding non-degenerate complex character of U. Isomorphism classes of quasi-split forms of D_4 are also parametrized by étale cubic algebras over F. For any such E we denote by H_E the corresponding quasi-split form of D_4.

In [7, Theorem 3.1] it is shown that any cuspidal representation of G_2 supports at least one non-degenerate Fourier coefficient along this unipotent subgroup. The integral $\mathcal{Z}_E\left(\chi, s, \varphi, f\right)$ involves a normalized degenerate Eisenstein series $\mathcal{E}_E^*\left(\chi, s, f, g\right)$ associated to the induced representation from the Heisenberg parabolic subgroup P_E of H_E.

The main result of this paper is

Theorem 3.1 . *Let π be an irreducible cuspidal representation of $G_2\left(\mathbb{A}_F\right)$ supporting the Fourier coefficient corresponding to E and let $\varphi = \bigotimes_{\nu \in \mathcal{P}} \varphi_\nu \in \pi$ and $f_s = \bigotimes_{\nu \in \mathcal{P}} f_\nu \in I_{P_E}\left(\chi, s\right)$ be factorizable data. Fix a finite set of places $S \subset \mathcal{P}$ so that for $\nu \notin S$ all data is unramified. Let*

$$\mathcal{Z}_E\left(\chi, s, \varphi, f\right) = \int\limits_{G_2(F)\backslash G_2(\mathbb{A}_F)} \varphi\left(g\right) \mathcal{E}_E^*\left(\chi, s, f, g\right) dg \tag{1.1}$$

then

$$\mathcal{Z}_E\left(\chi, s, \varphi, f\right) = \mathcal{L}^S\left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st}\right) d_S\left(\chi, s, \Psi_E, \varphi_S, f_S\right). \tag{1.2}$$

Moreover, for any s_0 there exist vectors φ_S, f_S such that $d_S\left(\chi, s, \Psi_E, \varphi_S, f_S\right)$ is analytic in a neighborhood of s_0 and $d_S\left(\chi, s_0, \Psi_E, \varphi_S, f_S\right) \neq 0$.

In particular, the family of twisted partial $\mathcal{L}$-function $\mathcal{L}^S\left(s, \pi, \chi, \mathfrak{st}\right)$ admits a meromorphic continuation to the whole complex plane.

This result generalizes [12, theorem 3.1] where we constructed an integral representation for the standard (untwisted) $\mathcal{L}$-function supporting the Fourier coefficient associated with the split cubic algebra over F.

The foremost challenge in the proof of Theorem 3.1 is that in the unfolded integral appears a non-unique model, namely a functional from $\mathrm{Hom}_{U(\mathbb{A}_F)}\left(\pi, \mathbb{C}_{\Psi_E}\right)$ which for many π-s is not a one dimensional space. Usually such integrals, called *new way integrals* after [15], are not factorizable. In [15] I. Piatetski-Shapiro and S. Rallis suggested a method to prove factorizability of such integrals.

The remarkable mechanism suggested there relies on the fact that the local integral at an unramified place ν will be equal to the local $\mathcal{L}$-factor **for any** functional from $\mathrm{Hom}_{U(F_\nu)}(\pi_\nu, \mathbb{C}_{\Psi_E})$.

As a byproduct of the proof of Theorem 3.1 we provide a parametrization of the $G_2(F)$ orbits in $P_E(F) \backslash H_E(F)$.

The proof of Theorem 3.1 is mostly unified for all étale cubic algebras E over F and the proof splits up for the different kinds of cubic algebras only at a very late stage, namely appendix A.2 and appendix B, where the arithmetical difference between the different étale algebras become critical.

In the proof of the local unramified identity we use the same strategy as in [12] and use an approximation to the generating function of the local $\mathcal{L}$-factor instead of the generating function itself. In section 6 we strengthen the results of [12] concerning this approximation.

As an application of the main theorem we deduce the following result characterizing theta lifts from groups of finite type as the representations having a certain pole. Let E be an étale cubic algebra which is not a non-Galois field extension. Let χ_E be the character attached to E/F by class field theory and let $S_E = \mathrm{Aut}_F(E)$. We denote by θ_{S_E} the theta correspondence for the dual pair $S_E \times G_2$ in $H_E \rtimes S_E$.

Theorem 8.1 . *The following are equivalent*

(1) $\mathcal{L}^S(s, \pi, \chi_E, \mathfrak{st})$ *admits a pole at $s = 2$ of order 2 if $E = F \times F \times F$ and of order 1 otherwise.*
(2) $\theta_{S_E}(\pi) \neq 0$.

Other Rankin-Selberg integrals representing the standard $\mathcal{L}$-function of cuspidal representations of G_2 were introduced in [9] for generic representations and in [10] for any cuspidal representation of G_2. The later is done using a doubling construction showing that the set of poles of $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$ are contained in the set of poles of a degenerate Eisenstein series of the exceptional group of type E_8. D. Ginzburg and J. Hundley further conjectured [10, Conjecture 1] for $\mathfrak{Re}(s) \geq 0$ the orders of the poles are bounded by 2.

2. **Preliminaries**

Let F be a number field and let $\mathcal{P}$ be its set of places. For any $\nu \in \mathcal{P}$ we denote by F_ν the completion of F at ν. If $\nu < \infty$ we denote by $\mathcal{O}_\nu$ its ring of integers, by ϖ_ν a uniformizer of F_ν and by q_ν the cardinality of the residue field of F_ν. We also denote by $\mathbb{A}_F = \mathbb{A}$ the ring of adeles of F.

2.1. **The Group G_2.** Let G be the simple split reductive adjoint and simply connected group of type G_2 defined over F. Let B be a Borel subgroup of G and T a maximal torus in B. Let α and β be the short and long simple roots of G with respect to (B, T). Let W be the Weyl group of G with respect to (B, T). The Dynkin diagram of G is

$$\overset{\alpha}{\circ}\!\!\Longleftarrow\!\!\overset{\beta}{\circ}.$$

The set of positive roots of G is

$$\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$

The fundamental weights of G are denoted by

$$\omega_1 = 2\alpha + \beta, \quad \omega_2 = 3\alpha + 2\beta.$$

For any simple root γ let $w_\gamma \in W$ be the simple reflection with respect to it. For any root γ we fix a one-parametric subgroup $x_\gamma : \mathbb{G}_a \to G$. Also, let $h_\gamma : \mathbb{G}_m \to T$ be the coroot subgroup such that for any root ϵ

$$\epsilon\left(h_\gamma\left(t\right)\right) = t^{\left\langle \epsilon, \gamma^\vee \right\rangle}.$$

The group G contains an Heisenberg maximal parabolic subgroup $P = M \cdot U$. The Levi subgroup M is isomorphic to GL_2 and is generated by the simple root α, while U is a five-dimensional Heisenberg group. Finally, we let $\mathfrak{st} : G \hookrightarrow GL_7$ be the standard 7-dimensional embedding.

2.2. Twisted Partial $\mathcal{L}$-functions.

The dual Langlands group ${}^L G$ of G is isomorphic to $G_2\left(\mathbb{C}\right)$.

Let $\pi = \underset{\nu \in \mathcal{P}}{\otimes} \pi_\nu$ be an irreducible cuspidal representation of $G\left(\mathbb{A}\right)$ and let $\chi = \underset{\nu \in \mathcal{P}}{\otimes} \chi_\nu : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a Hecke character, both unramified outside of a finite subset $S \subset \mathcal{P}$. For $\nu \notin S$ we denote its Satake parameter by t_{π_ν}. We let

$$\mathcal{L}^S\left(s, \pi, \chi, \mathfrak{st}\right) = \prod_{\nu \notin S} \frac{1}{\det\left(I - \mathfrak{st}\left(t_{\pi_\nu}\right)\chi\left(\varpi_\nu\right)q_\nu^{-s}\right)}.$$

This product converges for $\mathfrak{Re}\left(s\right) >> 0$ to an analytic function. In this paper we prove that $\mathcal{L}^S\left(s, \pi, \chi, \mathfrak{st}\right)$ admits meromorphic continuation to the whole complex plane.

Remark 2.1. The dual Langlands group ${}^L GL_1$ of GL_1 is isomorphic to $GL_1\left(\mathbb{C}\right) = \mathbb{C}^\times$. By abuse of notations by $\mathfrak{st}$ we also denote the irreducible seven-dimensional complex representation of $G_2\left(\mathbb{C}\right) \times \mathbb{C}^\times$. Since for $\nu \notin S$ the Satake parameter of χ_ν is $\chi_\nu\left(\varpi_\nu\right) \in \mathbb{C}^\times$ it holds that

$$\mathcal{L}^S\left(s, \pi, \chi, \mathfrak{st}\right) = \prod_{\nu \notin S} \frac{1}{\det\left(I - \mathfrak{st}\left(t_{\pi_\nu}, \chi\left(\varpi_\nu\right)\right)q_\nu^{-s}\right)}.$$

2.3. Étale Cubic Algebras Over F.

Let E be an étale cubic algebra over F, then E is one of the following:

(1) $\underline{E = F \times F \times F}$: This is called the split cubic algebra over F. In this case, $\mathrm{Aut}_F\left(E\right) \equiv S_3$. For $\left(a, b, c\right) \in F \times F \times F$ let $\mathrm{Nm}_{E/F}\left(a, b, c\right) = abc$.

(2) $\underline{E = F \times K}$: Here K is a quadratic (and hence Galois) extension of F. Furthermore, $\mathrm{Aut}_F\left(E\right) = \mathrm{Gal}\left(K/F\right) = \{1, \sigma\}$. For $\left(a, b\right) \in F \times K$ let $\mathrm{Nm}_{E/F}\left(a, b\right) = a\,\mathrm{Nm}_{K/F}\left(b\right) = abb^\sigma$.

(3) $\underline{E \text{ is a Cubic Galois Field Extension}}$: Here we assume that E is a cubic Galois extension of F. In this case, $\mathrm{Aut}_F\left(E\right) = \mathrm{Gal}\left(E/F\right) = \left\{1, \sigma, \sigma^2\right\}$ For $a \in E$ let $\mathrm{Nm}_{E/F}\left(a\right) = aa^\sigma a^{\sigma^2}$.

(4) $\underline{E \text{ is a Cubic non-Galois Field Extension}}$: In this case, let L be the Galois closure of E over F, this is a Sextic Galois extension with $\mathrm{Gal}\left(L/F\right) = \left\langle \sigma, \tau : \sigma^3 = 1, \tau^2 = 1 \right\rangle$. Note that L is also a Galois extension of E. We achieve the following tower of extensions
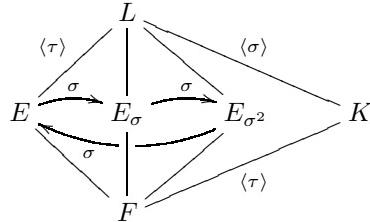
Where $K = L^{\langle \sigma \rangle}$ and $E, E_\sigma = L^{\langle \sigma\tau\sigma^2 \rangle}$ and $E_{\sigma^2} = L^{\langle \sigma^2\tau\sigma \rangle}$ are the σ-conjugates of E in L. For $a \in E$ let $\mathrm{Nm}_{E/F}\left(a\right) = aa^\sigma a^{\sigma^2}$.

Remark 2.2. We call the first three types **Galois étale cubic algebras over** F.

For reasons that will become clear later in this section, we seek a certain realization of E as a quotient of the polynomial ring in two variables $F[x, y]$. Let E be an étale cubic algebra, then one can choose a homogeneous cubic polynomial $p_E \in F[x, y]$ so that $F[x]/(p_E(x, 1)) \cong E$. Note that p_E splits in a finite extension of F into

$$p_E(x, y) = (x - ay)(x - by)(x - cy),$$

So one may choose p_E so that

$$p_E(x, y) = x^3 - T_{(a,b,c)}x^2y + D_{(a,b,c)}xy^2 - N_{(a,b,c)}y^3,$$

where $(a, b, c) \in \overline{F} \times \overline{F} \times \overline{F}$ and

- $T_{(a,b,c)} = a + b + c \in F$.
- $D_{(a,b,c)} = ab + bc + ca \in F$.
- $N_{(a,b,c)} = abc \in F$.

According to [18, Proposition 2.2], for any E one can choose (a, b, c) satisfying the following condition:

(CT): (a, b, c) is one of the following:

- $\underline{E = F \times F \times F}$: $(a, b, c) = (1, -1, 0)$.
- $\underline{E = F \times K, K \text{ is a field}}$: Choose $\theta \in K$ such that $K = F[\theta]$ and $\theta + \theta^\sigma = 0$. We choose $(a, b, c) = (0, \theta, \theta^\sigma) \in K \times K \times K$.
- $\underline{E \text{ is a field}}$: Choose $\theta \in E$ such that $E = F[\theta]$ and $\theta + \theta^\sigma + \theta^{\sigma^2} = 0$. We choose $(a, b, c) = \left(\theta, \theta^\sigma, \theta^{\sigma^2}\right) \in E \times E \times E$.

Here **CT** stands for *cubic algebra triplets*. We henceforth assume that (a, b, c) satisfy **(CT)**. In particular, we assume that $T_{(a,b,c)} = 0$.

2.4. **Characters of U.** In this subsection, we will construct for any E a character $\Psi_E \in \mathrm{Hom}\left(U(\mathbb{A}), \mathbb{C}^\times\right)$. We parametrize the elements of U by

$$u(r_1, r_2, r_3, r_4, r_5) := x_\beta(r_1) \, x_{\alpha+\beta}(r_2) \, x_{2\alpha+\beta}(r_3) \, x_{3\alpha+\beta}(r_4) \, x_{3\alpha+2\beta}(r_5).$$

The natural action of M on U induces an action on $\mathrm{Hom}(U, \mathbb{G}_a)$. It is shown in [13] that for any field L of characteristic 0, the $M(L)$-orbits in $\mathrm{Hom}(U(L), L)$ are naturally parametrized by isomorphism classes of cubic L-algebras.

For the rest of this paper, fix a non-trivial additive complex unitary character $\psi = \underset{\nu \in \mathcal{P}}{\otimes} \psi_\nu$ of $F \backslash \mathbb{A}$.

This gives rise to the correspondence

$$\left\{\begin{array}{c} \text{Isomorphism classes of} \\ \text{cubic algebras over } F \end{array}\right\} \longleftrightarrow \mathrm{Hom}\left(U(\mathbb{A}), \mathbb{C}^\times\right)/M(F).$$

In particular, we call $\Psi \in \mathrm{Hom}\left(U(\mathbb{A}), \mathbb{C}^\times\right)/M(F)$ *non-degenerate* if it corresponds to an étale cubic algebra over F.

For any E we will now attach a representative Ψ_E of the corresponding $M(F)$-orbit in $\mathrm{Hom}\left(U(\mathbb{A}), \mathbb{C}^\times\right)$. We let

$$(2.1) \qquad \Psi_E\left(u(r_1, r_2, r_3, r_4, r_5)\right) := \psi\left(r_4 - T_{(a,b,c)}r_3 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1\right).$$

In [12] we use a distinguished representative of the class of complex characters associated with the split cubic algebra over F, we denote it by Ψ_s. In section 4 we show the relation between Ψ_s and $\Psi_{F \times F \times F}$.

2.5. **Fourier Coefficients and Wave Front of a Representation.** We denote by $\mathcal{A}(G)$ the space of automorphic forms on $G(\mathbb{A})$. For any $\varphi \in \mathcal{A}(G)$ and any $\Psi \in \mathrm{Hom}(U(\mathbb{A}), \mathbb{C}^{\times})$ we let

$$L_{\Psi}(\varphi)(g) = \int\limits_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\, \overline{\Psi(u)} du.$$

For any $g \in G(\mathbb{A})$ this defines a functional $L_{\Psi}(\cdot)(g) \in \mathrm{Hom}_{U(\mathbb{A})}(\mathcal{A}(G), \mathbb{C}_{\Psi})$. For an automorphic representation π in $\mathcal{A}(G)$, we say that π supports the (U, Ψ)-Fourier coefficient if there exist $\varphi \in \pi$ so that $L_{\Psi}(\varphi) \not\equiv 0$. It was shown in [7] that for any cuspidal representation π in $\mathcal{A}(G)$, there exists an étale cubic algebra E so that π supports the Fourier coefficient corresponding to E. Conversely, it is shown in [8] that for any étale cubic algebra E there exists a cuspidal representation π that supports the (U, Ψ_E)-Fourier coefficient.

2.6. **Local Fourier Transform.** For a finite $\nu \in \mathcal{P}$, let K_{ν} denote the maximal compact subgroup $G(\mathcal{O}_{\nu})$ of $G(F_{\nu})$. Given a complex character Ψ of $U(F_{\nu})$ let

$$\mathcal{M}_{\Psi} = \left\{ f : G(F_{\nu}) \to \mathbb{C} : f(ugk) = \overline{\Psi(u)} f(g) \ \forall u \in U(F_{\nu}), \ k \in K_{\nu} \right\}.$$

We let $\mathcal{H}_{\nu} = \mathcal{H}(G(F_{\nu}), K_{\nu})$ denote the spherical Hecke algebra of $G(F_{\nu})$ with respect to K_{ν}. For $f \in \mathcal{H}_{\nu}$ define its Ψ-Fourier transform f^{Ψ} by

$$f^{\Psi}(g) = \int\limits_{U(F_{\nu})} f(ug)\, \Psi(u)\, du.$$

Obviously $f^{\Psi} \in \mathcal{M}_{\Psi}$.

Remark 2.3. For any $f \in \mathcal{M}_{\Psi}$, f is determined by its values on $M(F) \cap B(F)$. For any $g = h_{\alpha}(t_1)h_{\beta}(t_2)x_{\alpha}(d) \in M(F) \cap B(F)$ we may also write $g = x_{\alpha}(p)\, h_{\alpha}(t_1)h_{\beta}(t_2)$ with $p = \frac{dt_1^2}{t_2}$. The two different presentations will prove useful later.

The following lemma will be useful while evaluating functions in $\mathcal{M}_{\Psi_E}$, the proof is analogues to [12, Lemma A.1].

Lemma 2.1. Let $f \in \mathcal{M}_{\Psi_E}$, then for $f(h_{\alpha}(t_1)h_{\beta}(t_2)x_{\alpha}(d)) = 0$ unless the following holds:

$$N_E \frac{t_2^2}{t_1^3} + D_E \frac{dt_2}{t_1} + \frac{d^3 t_1^3}{t_2}, D_E \frac{t_2}{t_1} + \frac{3d^2 t_1^3}{2t_2}, \frac{3dt_1^3}{t_2}, \frac{t_1^3}{t_2} \in \mathcal{O}.$$

2.7. **Quasi-Split Forms of** D_4. We recall the following parametrization of quasi-split forms of D_4 over F:

$$\{\text{Quasi-split forms of } D_4 \text{ over } F\} \longleftrightarrow \left\{\varphi : \mathrm{Gal}\left(\bar{F}/F\right) \to \mathrm{Aut}\left(Dyn\left(D_4\right)\right)\right\},$$

where $Dyn(D_4)$ is the Dynkin diagram of type D_4.

$$
\begin{array}{ccc}
\alpha_1 & \alpha_2 & \alpha_3 \\
\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\bigcirc & &
\end{array}
$$

with α_4 below α_2.
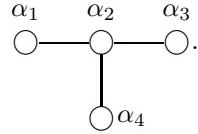
Since $\mathrm{Aut}(Dyn(D_4)) \cong S_3$ we have

$$\{\text{Quasi-split forms of } D_4 \text{ over } F\} \longleftrightarrow \left\{\varphi : \mathrm{Gal}\left(\bar{F}/F\right) \to S_3\right\}.$$

On the other hand, there is a bijection

$$\left\{\varphi : \mathrm{Gal}\left(\bar{F}/F\right) \to S_3\right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{étale cubic algebras over } F \end{array} \right\}.$$

For any cubic algebra E let $S_E = Aut_F(E)$ which is a twisted form of S_3. An action of S_E on the algebraic group $Spin_8$ determines a simply-connected quasi-split form $H_E = D_4^E$ of D_4 over F. We fix a Chevalley-Steinberg system of épinglage [3, 4.1.3]

$$\left\{ T_E, B_E, x_\gamma : \mathbb{G}_a \to (H_E)_\gamma, \gamma \in \Phi_{D_4} \right\},$$

where $T_E \subset B_E$ is a maximal torus contained in a Borel subgroup (both defined over F) and Φ_{D_4} are the roots of $H_E \otimes \overline{F} \equiv D_4(\overline{F})$. For any γ in the reduced root system of H_E we denote by F_γ the field of definition of γ.

We now give a more detailed description of H_E for the different kinds of étale cubic algebras over F in terms of the action of $\mathrm{Gal}\left(\overline{F}/F\right)$ on $H_E\left(\overline{F}\right)$.

(1) $\underline{E = F \times F \times F}$: In this case H_E is the split reductive simply-connected group of type D_4 over F. It corresponds to the trivial action of $\mathrm{Gal}\left(\bar{F}/F\right)$.

(2) $\underline{E = F \times K}$: In this case we make a choice of one of the roots $\alpha_1, \alpha_3, \alpha_4$ and glue the other two together. The fact that for each choice of distinct root we get an isomorphic algebraic group is called *triality*. In what follows we choose α_1 to be the distinct root. This is the case where $E = F \times K$ with K a quadratic (and hence Galois) extension of F. It is enough to define an action of $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ on $Spin_8(K)$. This action is determined by

$$\sigma(x_{\alpha_1}(k)) = x_{\alpha_1}(\sigma(k))$$
$$\sigma(x_{\alpha_2}(k)) = x_{\alpha_2}(\sigma(k))$$
$$\sigma(x_{\alpha_3}(k)) = x_{\alpha_4}(\sigma(k))$$
$$\sigma(x_{\alpha_4}(k)) = x_{\alpha_3}(\sigma(k)).$$

(3) $\underline{E \text{ is a Cubic Galois Field Extension}}$: Here we assume that E is a cubic Galois extension of F. It is enough to define an action of $\mathrm{Gal}(E/F) = \langle \sigma : \sigma^3 = 1 \rangle$ on $Spin_8(E)$. This action is determined by

$$\sigma(x_{\alpha_2}(e)) = x_{\alpha_2}(\sigma(e))$$
$$\sigma(x_{\alpha_1}(e)) = x_{\alpha_3}(\sigma(e))$$
$$\sigma(x_{\alpha_3}(e)) = x_{\alpha_4}(\sigma(e))$$
$$\sigma(x_{\alpha_4}(e)) = x_{\alpha_1}(\sigma(e)).$$

(4) $\underline{E \text{ is a Cubic non-Galois Field Extension}}$: Here we assume that E is a cubic non-Galois extension of F. In order to define $H_E(F)$ we first consider the Galois closure L of E over F as above with $\mathrm{Gal}(L/F) = \langle \sigma, \tau : \sigma^3 = 1, \tau^2 = 1 \rangle$.

The action of Γ_E on $Spin_8(L)$ is determined by

$$\tau(x_{\alpha_i}(l)) = x_{\alpha_i}(\tau(l)) \ \forall i$$
$$\sigma(x_{\alpha_2}(l)) = x_{\alpha_2}(\sigma(l))$$
$$\sigma(x_{\alpha_1}(l)) = x_{\alpha_3}(\sigma(l))$$
$$\sigma(x_{\alpha_3}(l)) = x_{\alpha_4}(\sigma(l))$$
$$\sigma(x_{\alpha_4}(l)) = x_{\alpha_1}(\sigma(l)).$$

Also, we have $B_E \cap G = B$. Furthermore, there exists an Heisenberg parabolic subgroup $P_E = M_E \cdot U_E$ of H_E so that $P_E \cap G = P$, $M_E \cap G = M$ and $U_E \cap G = U$. In particular

$$M_E \equiv \left\{ g \in \mathrm{Res}_{E/F} \, \mathrm{GL}_2 : \det(g) \in \mathbb{G}_m \right\}.$$

2.8. **The Degenerate Eisenstein Series.** Fix a Hecke character $\chi : \mathbb{A}^\times \to \mathbb{C}^\times$. We consider the normalized induction

$$I_{P_E}(\chi, s) = \text{Ind}_{P_E}^{H_E}(\chi \circ \det_{M_E}) |\det_{M_E}|^{s + \frac{5}{2}},$$

where $\det_{M_E}$ is the determinant character associated to the Levi subgroup M_E. Note that for the modulus character of P_E it holds that $\delta_{P_E}|_{M_E} = |\det_{M_E}|^5$.

For any K-finite standard section $f_s \in I_{P_E}(\chi, s)$ we define the following degenerate Eisenstein series

$$(2.2) \qquad \mathcal{E}_E(\chi, f_s, s, g) = \sum_{\gamma \in P_E(F) \backslash H_E(F)} f_s(\gamma g).$$

This series converges for $\mathfrak{Re}(s) >> 0$ and admits a meromorphic continuation to the whole complex plane. We normalize the Eisenstein series as follows

$$\mathcal{E}_E^*(\chi, f_s, s, g) = j_E(\chi, s)\, \mathcal{E}_E(\chi, f_s, s, g),$$

where

$$j_E(\chi, s) = \begin{cases} \mathcal{L}_F\left(s + \frac{5}{2}, \chi\right) \mathcal{L}_F\left(s + \frac{3}{2}, \chi\right)^2 \mathcal{L}_F\left(2s + 1, \chi^2\right), & E = F \times F \times F \\ \mathcal{L}_F\left(s + \frac{5}{2}, \chi\right) \mathcal{L}_K\left(s + \frac{3}{2}, \chi \circ \text{Nm}_{E/F}\right) \mathcal{L}_F\left(2s + 1, \chi \circ \text{Nm}\right), & E = F \times K \\ \dfrac{\mathcal{L}_F\left(s + \frac{5}{2}, \chi\right) \mathcal{L}_E\left(s + \frac{3}{2}, \chi \circ \text{Nm}_{E/F}\right) \mathcal{L}_F\left(2s + 1, \chi^2\right)}{\mathcal{L}_F\left(s + \frac{3}{2}, \chi\right)}, & E \text{ field} \end{cases}.$$

3. The Zeta Integral

Let $\pi = \underset{\nu \in \mathcal{P}}{\otimes} \pi_\nu$ be an irreducible cuspidal representation supporting an (U, Ψ_E)-Fourier coefficient for some étale cubic algebra E over F. Fix a corresponding character $\Psi_E = \underset{\nu \in \mathcal{P}}{\otimes} \Psi_{(a,b,c),\nu}$ of $U(\mathbb{A})$ where (a, b, c) satisfy **(CT)**. Let $\chi = \underset{\nu \in \mathcal{P}}{\otimes} \chi_\nu$ be a Hecke character. For $\varphi \in \pi$ and a standard section $f_s \in I_{P_E}(\chi, s)$ we consider the following integral

$$(3.1) \qquad \mathcal{Z}_E(\chi, s, \varphi, f) = \int_{G(F) \backslash G(\mathbb{A})} \varphi(g)\, \mathcal{E}_E^*(\chi, s, f, g)\, dg.$$

Since φ is cuspidal, and hence rapidly decreasing, this integral defines a meromorphic function in the complex plane. Hence the meromorphic continuation of $\mathcal{Z}_E(\chi, s, \varphi, f)$ follows from that of $\mathcal{E}_E^*(\chi, s, f, g)$.

Let $\varphi = \underset{\nu \in \mathcal{P}}{\otimes} \varphi_\nu \in \pi$ and $f_s = \underset{\nu \in \mathcal{P}}{\otimes} f_\nu \in I_{P_E}(\chi, s)$ be pure tensor products. Let $S \subset \mathcal{P}$ be a finite set of places such that for $\nu \notin S$ it holds that

- $2, 3 \nmid \nu$ and $\nu \neq \infty$.
- E_ν is unramified over F_ν.
- π_ν and χ_ν are unramified.
- φ_ν and f_ν are spherical.
- ψ_ν is of conductor $\mathcal{O}_\nu$.
- Either $a = 0$ or $|a| \in \mathcal{O}_\nu^\times$ and similarly for b and c.
- Either $D_{a,b,c} = 0$ or $D_{a,b,c} \in \mathcal{O}_\nu^\times$ and similarly for $N_{a,b,c}$.

Remark 3.1. Note that if $N_{(a,b,c)} = 0$ then necessarily $D_{(a,b,c)} \neq 0$.

Our main result is

Theorem 3.1. *Let π, φ, f and $S \subset \mathcal{P}$ be as above. Then*

$$(3.2) \qquad \mathcal{Z}_E\left(\chi, s, \varphi, f\right) = \mathcal{L}^S\left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st}\right) d_S\left(\chi, s, \Psi_E, \varphi_S, f_S\right)$$

Moreover, for any s_0 there exist vectors φ_S, f_S such that $d_S\left(\chi, s, \Psi_E, \varphi_S, f_S\right)$ is analytic in a neighborhood of s_0 and $d_S\left(\chi, s_0, \Psi_E, \varphi_S, f_S\right) \neq 0$.

In particular, the family of twisted partial $\mathcal{L}$-function $\mathcal{L}^S\left(s, \pi, \chi, \mathfrak{st}\right)$ admits a meromorphic continuation to the whole complex plane.

Most of this paper is devoted to the proof of this theorem. In this section we will outline the main ideas and defer the technical part to later sections and appendices. In this paper we generalize the approach presented in [12] while dealing with some delicate issues arising in the non-split case while, as noted in the introduction, do this uniformly for most of the way. The only place where the proofs differ between the different étale cubic algebras is at appendix A.2 and appendix B where the difference is caused by an arithmetic difference.

An application of this theorem to the θ-lift for the dual pairs (G, S_E) is discussed in section 8.

In order to perform the unfolding of this integral, we describe the $G(F)$-orbits in $P_E(F) \backslash H_E(F)$, this is done in section 4. In section 5 we prove the following theorem.

Theorem 3.2 (Unfolding). *For $\mathfrak{Re}(s) \gg 0$ it holds that*

$$(3.3) \qquad \mathcal{Z}_E\left(\chi, s, \varphi, f\right) = \int\limits_{U(\mathbb{A}) \backslash G(\mathbb{A})} L_{\Psi_E}\left(\varphi\right)\left(g\right) F^*\left(\Psi_E, \chi, g, s\right),$$

where

$$(3.4) \qquad F^*\left(\Psi_E, \chi, g, s\right) = \int\limits_{\mathbb{A}} f_s\left(\mu_E x_{3\alpha+\beta}\left(r\right) g\right) \psi\left(r\right) dr$$

and

$$\mu_E = w_2 w_1 w_3 w_4 x_{\alpha_1}\left(a\right) x_{\alpha_3}\left(b\right) x_{\alpha_4}\left(c\right).$$

Remark 3.2. Note that if the section f is factorizable then so is F^*. In particular

$$F^*\left(\Psi_E, \chi, g, s\right) = \prod\limits_{\nu \in \mathcal{P}} F^*_\nu\left(\Psi_{E,\nu}, \chi_\nu, g_\nu, s\right),$$

where

$$F^*_\nu\left(\Psi_{E,\nu}, \chi_\nu, g_\nu, s\right) = \int\limits_{F_\nu} f_s\left(\mu_E x_{3\alpha+\beta}\left(r\right) g_\nu\right) \psi_\nu\left(r\right) dr.$$

On the other hand, the integral in equation (3.3) is a priori not factorizable since the space $\mathrm{Hom}_{U(\mathbb{A})}\left(\pi, \mathbb{C}_\Psi\right)$ is not necessarily one-dimensional and may even be infinite dimensional. Nevertheless, we will show that the integral is factorizable after all, this follows from an inductive process suggested in [15] and the following two results.

Theorem 3.3 (Unramified Computation). *Let π_ν be an irreducible unramified representation of $G(F_\nu)$, and let v_0 be a fixed spherical vector in π_ν. There exists $s_0 \in \mathbb{R}$ such that for any $\mathfrak{Re}(s) > s_0$ and **any** $\Lambda \in \mathrm{Hom}_{U(F_\nu)}\left(\pi_\nu, \mathbb{C}_{\Psi_{E,\nu}}\right)$ it holds that*

$$(3.5) \qquad \int\limits_{U(F_\nu) \backslash G(F_\nu)} F^*_\nu\left(\Psi_{E,\nu}, \chi_\nu, g, s\right) \Lambda\left(\pi_\nu\left(g\right) v_0\right) dg = \mathcal{L}\left(s + \frac{1}{2}, \pi_\nu, \chi_\nu, \mathfrak{st}\right) \Lambda\left(v_0\right).$$

Remark 3.3. The case of a split Fourier coefficient and $\chi \equiv 1$ is dealt in [12]. The function F^* has a different form there due to the different choice of representative for the open orbit that gives rise to a different character Ψ_s in the same orbit of $\Psi_{F \times F \times F}$. In section 6 we relate $F_\nu^* (\Psi_{E,\nu}, \chi_\nu, g, s)$ and $F_\nu^* (\Psi_{s,\nu}, \chi_\nu, g, s)$.

Theorem 3.4 (Ramified Computation)**.** *For any $s_0 \in \mathbb{C}$ there exist datum φ_S and f_S such that $d_S (\chi, s, \Psi_E, \varphi_S, f_S)$ is holomorphic and non-vanishing in a neighborhood of s_0.*

3.3 is proved in section 6 together with the appendices, Theorem 3.4 is proved in section 7. We now show how to derive the main theorem from the other results presented in this section. The proof is essentially the same as in [12], we include it for the convenience of the reader.

Proof of Theorem 3.1. By Theorem 3.2

$$(3.6) \qquad \mathcal{Z}_E (\chi, s, \varphi, f) = \lim_{\substack{\longrightarrow \\ S \subset \Omega \subset \mathcal{P} \\ |\Omega| < \infty}} \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} L_{\Psi_s} (\varphi) (g) F_\Omega^* \left(\Psi_{(a,b,c),\Omega}, \chi_\Omega, g, s \right) dg ,$$

where $G(\mathbb{A})_\Omega = \prod_{\nu \in \Omega} G(F_\nu)$ and

$$F_\Omega^* (\Psi_{E,\Omega}, \chi_\Omega, g, s) = j_{E,\Omega}(s) \int_{F_\Omega} f_s \left(\mu_E x_{3\alpha+\beta} (r) g \right) \psi_\Omega (r) \, dr.$$

Fix $s_0 \in \mathbb{R}$ such that the right hand side of equation (3.3) converges for $\mathfrak{Re}\, s > s_0$. The integrals on the right hand side of equation (3.6) must also converge there. Fix a finite subset $S \subseteq \Omega \subset \mathcal{P}$ and $\nu \notin \Omega$. Also fix $s_1 \in \mathbb{R}$ such that Theorem 3.3 holds for $\mathfrak{Re}\, s > s_1$ and π_ν. It the holds that

$$\int_{U(\mathbb{A})_{\Omega \cup \{\nu\}} \backslash G(\mathbb{A})_{\Omega \cup \{\nu\}}} L_{\Psi_E} (\varphi) (g) F_{\Omega \cup \{\nu\}}^* \left(\Psi_{E,\Omega \cup \{\nu\}}, \chi_{\Omega \cup \{\nu\}}, g, s \right) dg =$$

$$\int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} \int_{U(F_\nu) \backslash G(F_\nu)} L_{\Psi_E} (\varphi) (gg_\nu) F_{\Omega \cup \{\nu\}}^* \left(\Psi_{E,\Omega \cup \{\nu\}}, \chi_{\Omega \cup \{\nu\}}, gg_\nu, s \right) dg_\nu \, dg =$$

$$\int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} F_\Omega^* (\Psi_{E,\Omega}, \chi_\Omega, g, s) \int_{U(F_\nu) \backslash G(F_\nu)} L_{\Psi_E} (\varphi) (gg_\nu) F_\nu^* (\Psi_{E,\nu}, \chi_\nu, g_\nu, s) \, dg_\nu \, dg =$$

$$\mathcal{L} \left(s + \frac{1}{2}, \pi_\nu, \mathfrak{st} \right) \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} L_{\Psi_E} (\varphi) (g) F_\Omega^* (\Psi_{E,\Omega}, \chi_\Omega, g, s) \, dg ,$$

where the last equality is due to Theorem 3.3. A priori the last equality holds only for $\mathfrak{Re}\, s > \max\{s_0, s_1\}$, but since $\mathcal{L} \left(s + \frac{1}{2}, \pi_\nu, \mathfrak{st} \right)$ is a meromorphic function the equality actually holds for $\mathfrak{Re}\, s > s_0$. Plugging this into equation (3.6) we get

$$\mathcal{Z}_E (\chi, s, \varphi, f) = \lim_{\substack{\longrightarrow \\ S \subset \Omega \subset \mathcal{P} \\ |\Omega| < \infty}} \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} L_{\Psi_E} (\varphi) (g) F_\Omega^* (\Psi_{E,\Omega}, \chi_\Omega, g, s) \, dg =$$

$$= \lim_{\substack{\longrightarrow \\ S \subset \Omega \subset \mathcal{P} \\ |\Omega| < \infty}} \prod_{\nu \in \Omega \backslash S} \mathcal{L} \left(s + \frac{1}{2}, \pi_\nu, \chi, \mathfrak{st} \right) \int_{U(\mathbb{A})_S \backslash G(\mathbb{A})_S} L_{\Psi_E} (\varphi) (g) F_S^* (\Psi_{E,S}, \chi_S, g, s) \, dg =$$

$$= \mathcal{L}^S \left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st} \right) \int_{U(\mathbb{A})_S \backslash G(\mathbb{A})_S} L_{\Psi_E} (\varphi) (g) F_S^* (\Psi_{E,S}, \chi_S, g, s) \, dg .$$

We finish the proof by fixing our datum according to Theorem 3.4and taking

$$d_S\left(s,\varphi_S,f_S\right) = \int_{U(\mathbb{A})_S \backslash G(\mathbb{A})_S} L_{\Psi_E}\left(\varphi\right)(g) F_S^*\left(\Psi_{E,S},\chi_S,g,s\right)\,dg\,.$$

<div align="right">□</div>

4. G_2 **Orbits in** $P_E \backslash H_E$

In order to perform the unfolding of equation (3.1) we need to describe $P_E\left(F\right)\backslash H_E\left(F\right)/G\left(F\right)$ for any étale cubic algebra E over F. In order to do this, we give a description of $\left(P_E\backslash H_E\right)\left(\overline{F}\right)/G\left(\overline{F}\right)$. Although $\left(P_E\backslash H_E\right)\left(\overline{F}\right)/G\left(\overline{F}\right)$ is independent of E, the representatives we choose depend on E so that we can use the method of Galois descent in order to compute $\left(P_E\backslash H_E\right)\left(F\right)/G\left(F\right)$.

Denote $P_E\backslash H_E$ by X_E, this is a projective variety with a right G-action. Note that $X_E\left(F\right) = \left(P_E\backslash H_E\right)\left(F\right) = P_E\left(F\right)\backslash H_E\left(F\right)$ due to [2, Theorem 4.13a]. Let $Q = L \cdot V$ be the non-Heisenberg maximal parabolic subgroup of G. Its Levi part $L \cong GL_2$ is generated by the root β. The unipotent radical V of Q is a three-step unipotent group, we denote its commutator $[V, V]$ by R. We recall from [14, Lemma 2.1] that $X_E\left(\overline{F}\right)$ has five $G\left(\overline{F}\right)$-orbits, given as follows.

Lemma 4.1. *The following is a list of representatives of the $G\left(\bar{F}\right)$-orbits in $X_E\left(\bar{F}\right)$ and their stabilizers:*

    (1) *$\mu = 1$ and the stabilizer of $P_E\left(\bar{F}\right)\mu G\left(\bar{F}\right)$ is $G^\mu = P$.*

    (2) *$\mu = w_2 w_1, w_2 w_3, w_2 w_4$ and the stabilizer of $P_E\left(\bar{F}\right)\mu G\left(\bar{F}\right)$ is $G^\mu = LR$.*

    (3) *$\mu = w_2 w_3 x_{-\alpha_1}\left(1\right)$ is a representative of the open orbit and the stabilizer of $P_E\left(\bar{F}\right)\mu G\left(\bar{F}\right)$ is $G^\mu = T_{3\alpha+2\beta}\cdot U^\mu$, where*

$$T_{3\alpha+2\beta} = \left\{h_{3\alpha+2\beta}\left(t\right) : t \in \bar{F}^\times\right\}, \quad U^\mu = \left\{u\left(r_1, r_2, r_2, r_4, r_5\right) : r_i \in \bar{F}\right\}$$

We now give a different set of representatives for these orbits. For a triple $(a, b, c) \in \overline{F} \times \overline{F} \times \overline{F} = \left(\mathrm{Res}_{E/F}\,\mathbb{G}_a\right)\left(\overline{F}\right)$ we denote

$$\mu\left(a, b, c\right) = w_2 w_1 w_3 w_4 x_{\alpha_1}\left(a\right) x_{\alpha_3}\left(b\right) x_{\alpha_4}\left(c\right).$$

For $x, x' \in X_E\left(\overline{F}\right)$ we say that $x \sim x'$ if they lie in the same $G\left(\overline{F}\right)$-orbit.

Lemma 4.2. *Let $(a, b, c) \in \overline{F} \times \overline{F} \times \overline{F}$ then*

    (1) *$a = b = c$ if and only if $\mu\left(a, b, c\right) \sim 1$*

    (2)   • *$a = b \neq c$ if and only if $\mu\left(a, b, c\right) \sim w_2 w_4$*

        • *$a \neq b = c$ if and only if $\mu\left(a, b, c\right) \sim w_2 w_1$*

        • *$a = c \neq b$ if and only if $\mu\left(a, b, c\right) \sim w_2 w_3$*

    (3) *a, b and c are distinct if and only if $\mu\left(a, b, c\right) \sim w_2 w_3 x_{-\alpha_1}\left(1\right)$*

Remark 4.1. The proof of this lemma relies on the use of Möbius transformations. As $M_E\left(\overline{F}\right) \cong \left(GL_2 \times GL_2 \times GL_2\right)^0\left(\overline{F}\right)$ we have a natural map

$$\left(GL_2 \times GL_2 \times GL_2\right)^0\left(\overline{F}\right) \overset{\Upsilon}{\to} P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)$$

$$m \mapsto P_E\left(\overline{F}\right) w_2\, m.$$

We also note that $w_2\left(M_E\left(\overline{F}\right) \cap B_E\left(\overline{F}\right)\right)w_2 \subset P_E\left(\overline{F}\right)$ and hence this map factors through

$$\left(B_0 \times B_0 \times B_0\right)^0\left(\overline{F}\right)\backslash\left(GL_2 \times GL_2 \times GL_2\right)^0\left(\overline{F}\right),$$

where B_0 denotes the Borel subgroup of GL_2. This quotient admits an action of GL_2 from the right diagonally. On the other hand, $M \cong GL_2$ acts from the right on $P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)$. The map Υ

is evidently GL_2-equivariant. Namely, for any $(m_1, m_2, m_3) \in (GL_2 \times GL_2 \times GL_2)^0 \left(\overline{F} \right)$ and any $m \in GL_2 \left(\overline{F} \right)$ it holds that

$$\Upsilon \left((m_1, m_2, m_3) \cdot m \right) = \Upsilon \left((m_1, m_2, m_3) \right) \cdot m$$

Recall that due to the Bruhat decomposition, $\left(B_0 \left(\overline{F} \right) \backslash GL_2 \left(\overline{F} \right) \right) \cong \mathbb{P}^1 \left(\overline{F} \right)$. This identification can be realized as follows

$$x \longleftrightarrow B_0 \left(\overline{F} \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \infty \longleftrightarrow B_0 \left(\overline{F} \right).$$

Hence, the right action of $M \left(\overline{F} \right)$ on $P_E \left(\overline{F} \right) \backslash H_E \left(\overline{F} \right)$ is induced from the action of $GL_2 \left(\overline{F} \right)$ on triples of points in $\mathbb{P}^1 \left(\overline{F} \right)$. In particular, using this projective coordinates, we note that $w_2 w_1 = \Upsilon (0, \infty, \infty)$, $w_2 w_3 = \Upsilon (\infty, 0, \infty)$, $w_2 w_4 = \Upsilon (\infty, \infty, 0)$ and $w_2 w_3 x_{-\alpha_1} (1) = \Upsilon (1, 0, \infty)$. As for the trivial orbit, 1 is not in the image of Υ but $1 \sim \Upsilon (\infty, \infty, \infty)$.

Furthermore, the orbits of $GL_2 \left(\overline{F} \right)$ in $\mathbb{P}^1 \left(\overline{F} \right) \times \mathbb{P}^1 \left(\overline{F} \right) \times \mathbb{P}^1 \left(\overline{F} \right)$ correspond to the items in Lemma 4.2. Namely, $\left(\mathbb{P}^1 \left(\overline{F} \right) \times \mathbb{P}^1 \left(\overline{F} \right) \times \mathbb{P}^1 \left(\overline{F} \right) \right) / GL_2 \left(\overline{F} \right)$ is given by:

(1) $\left\{ (a, a, a) : a \in \mathbb{P}^1 \left(\overline{F} \right) \right\}$.

(2) • $\left\{ (a, a, c) : a, c \in \mathbb{P}^1 \left(\overline{F} \right) \text{ distinct} \right\}$.
   • $\left\{ (a, b, b) : a, b \in \mathbb{P}^1 \left(\overline{F} \right) \text{ distinct} \right\}$.
   • $\left\{ (a, b, a) : a, b \in \mathbb{P}^1 \left(\overline{F} \right) \text{ distinct} \right\}$.

(3) $\left\{ (a, b, c) : a, b, c \in \mathbb{P}^1 \left(\overline{F} \right) \text{ distinct} \right\}$.

In fact, the lemma proves that there is a natural bijection

$$(4.1) \quad (B_0 \times B_0 \times B_0)^0 \left(\overline{F} \right) \backslash (GL_2 \times GL_2 \times GL_2)^0 \left(\overline{F} \right) / GL_2 \left(\overline{F} \right) \longleftrightarrow P_E \left(\overline{F} \right) \backslash H_E \left(\overline{F} \right) / G \left(\overline{F} \right).$$

Proof.   (1) In this case $\mu (a, a, a) = w_2 w_1 w_3 w_4 x_\alpha (a) \in G_2 \left(\overline{F} \right)$ and hence $\mu (a, a, a) \sim 1$.

(2) We prove for example that $\mu (a, a, c) \sim w_2 w_4$ for $a \neq c$. In this case, let

$$m (a, a, c) = x_\alpha (-a) w_\alpha \beta^\vee (a - c) x_\alpha (1) \in M (F) \subset G (F).$$

One checks that

$$\mu (a, a, c) \, m (a, a, c) \, (w_2 w_4)^{-1} \in B_E \left(\overline{F} \right) \subset P_E \left(\overline{F} \right)$$

and hence

$$\mu (a, a, c) \sim w_2 w_4.$$

(3) Let

$$m_{a,b,c} = \alpha^\vee (b - a) \, \beta^\vee \left(\frac{(a - b)^3}{(a - c)(b - c)} \right) x_\alpha \left(c \frac{(a - b)}{(c - a)(b - c)} \right) w_\alpha x_\alpha \left(\frac{(a - c)}{(a - b)} \right) \in M (F) \subset G (F).$$

One checks that

$$\mu (a, b, c) \, m_{a,b,c} \, (w_2 w_3 x_{-\alpha_1} (1))^{-1} \in B_E \left(\overline{F} \right) \subset P_E \left(\overline{F} \right),$$

hence

$$\mu (a, b, c) \sim w_2 w_3 x_{-\alpha_1} (1) \ .$$

Since Lemma 4.1 gives a list of representatives of orbits, the other direction follows immediately.

$\square$

For the rest of the paper we fix a triple (a, b, c) as in **(CT)** and write $\mu_E = \mu (a, b, c)$.

Corollary 4.1. $\mathrm{Stab}_{G \left(\overline{F} \right)} \left(P_E \left(\overline{F} \right) \mu_E \right) = T_{3\alpha + 2\beta} \cdot \ker \Psi_E$.

Proof. Recall from [12] that Ψ_s is given by

$$\Psi_s\left(u\left(r_1, r_2, r_3, r_4, r_5\right)\right) = \psi\left(r_2 + r_3\right).$$

and note that

$$\mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right) w_2 w_3 x_{-\alpha_1}\left(1\right)\right) = T_{3\alpha+2\beta} \cdot \ker \Psi_s.$$

We also note that $\ker \Psi_s$ and $\ker \Psi_E$ define algebraic groups over F.

From Lemma 4.2

$$P_E\left(\overline{F}\right) w_2 w_3 x_{-\alpha_1}\left(1\right) = P_E\left(\overline{F}\right) \mu_E m_{a,b,c}$$

and hence

$$\mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right) \mu_E\right) = m_{a,b,c}\,\mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right) w_2 w_3 x_{-\alpha_1}\left(1\right)\right) m_{a,b,c}^{-1}.$$

On the other hand,

$$\Psi_E\left(u\right) = \Psi_s\left(m_{a,b,c}^{-1} u m_{a,b,c}\right)$$

and hence

$$\ker \Psi_E = m_{a,b,c}^{-1}\left(\ker \Psi_s\right) m_{a,b,c}.$$

Also, $T_{3\alpha+2\beta} = \mathcal{Z}\left(M\right)$ and hence $m_{a,b,c}^{-1} T_{3\alpha+2\beta} m_{a,b,c} = T_{3\alpha+2\beta}$. $\qquad\square$

We now move to describing the $G\left(F\right)$ orbits in $X_E\left(F\right)$. Given $\mu \in H_E\left(F\right) \subseteq H_E\left(\overline{F}\right)$ we have a short exact sequence

$$\{1\} \to \mathrm{Stab}_{G(\bar{F})}\left(\mu\right) \to G_2\left(\bar{F}\right) \to P_E\left(\overline{F}\right) \mu G\left(\overline{F}\right) \to \{1\}$$

Denote by X_μ the G-orbit of $P_E\left(F\right)\mu \in X_E\left(F\right)$. We then have ([1, II.4.7]) a long exact sequence in the cohomology

In particular, there is a bijection (of sets)

$$X_\mu\left(F\right)/G_2\left(F\right) \longleftrightarrow \mathrm{Ker}\left[\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(F)}\left(\mu\right)\right) \to \mathrm{H}^1\left(\Gamma_E, G_2\left(F\right)\right)\right].$$

Since G_2 is a split and reductive algebraic group, then according to [17, §8] it holds that $\mathrm{H}^1\left(\Gamma_E, G_2\left(F\right)\right) = \{1\}$ and hence $X_\mu\left(F\right)/G_2\left(F\right)$ is parametrized by $\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(F)}\left(\mu\right)\right)$.

Theorem 4.1. *The $G\left(F\right)$-orbits in $P_E\left(F\right)\backslash H_E\left(F\right)$ are*

(1) $\underline{E = F \times F \times F}$: $P_E\left(F\right)$, $P_E\left(F\right) w_2 w_1$, $P_E\left(F\right) w_2 w_3$, $P_E\left(F\right) w_2 w_4$ *and* $P_E\left(F\right) w_2 w_1 w_3 w_4 x_{\alpha_1}\left(1\right) x_{\alpha_3}\left(-1\right)$.
(2) $\underline{E = F \times K,\ K = F\left[\theta\right]\ a\ field}$: $P_E\left(F\right)$, $P_E\left(F\right) w_2 w_1$ *and* $P_E\left(F\right) w_2 w_1 w_3 w_4 x_{\alpha_3}\left(\theta\right) x_{\alpha_4}\left(\theta^\sigma\right)$.
(3) $\underline{E = F\left[\theta\right]\ a\ field}$: $P_E\left(F\right)$ *and* $P_E\left(F\right) w_2 w_1 w_3 w_4 x_{\alpha_1}\left(\theta\right) x_{\alpha_3}\left(\theta^\sigma\right) x_{\alpha_4}\left(\theta^{\sigma^2}\right)$.

Proof. (1) This is exactly Lemma 4.1 combined with Lemma 4.2.
(2) We start by noting that $P_E\left(F\right)$, $P_E\left(F\right) w_2 w_4$ and $P_E\left(F\right) w_2 w_1 w_3 w_4 x_{\alpha_1}\left(\theta\right) x_{\alpha_3}\left(\theta^\sigma\right)$ are the only $G\left(\overline{F}\right)$-orbits in $P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)$ that intersects $H_E\left(F\right)$, we need only show that for these μ-s, $P_E\left(F\right)\mu$ is one $G\left(F\right)$-orbit.

- $\underline{\mu = 1}$: In this case, $\mathrm{Stab}_G(\mu) = P$ and we have a short exact sequence

$$\{1\} \to U \to P \to M \to \{1\}.$$

$\mathrm{H}^1(\Gamma_E, U(F)) = \{1\}$ since U is unipotent and $\mathrm{H}^1(\Gamma_E, M(F)) = \{1\}$ due to Hilbert 90' and thus $\mathrm{H}^1(\Gamma_E, P(F)) = \{1\}$. Hence $X_1(F)/G_2(F) = \{1\}$.

- $\underline{\mu = w_2 w_1}$: In this case, $\mathrm{Stab}_G(P_E \mu) = L \cdot R$ and we have a short exact sequence

$$\{1\} \to R \to \mathrm{Stab}_G(P_E \mu) \to L \to \{1\}.$$

$\mathrm{H}^1(\Gamma_E, R(F)) = \{1\}$ since R is unipotent and $\mathrm{H}^1(\Gamma_E, L(F)) = \{1\}$ due to Hilbert 90' and thus $\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(F)}(P_E(F)\mu)\right) = \{1\}$. Hence $X_\mu(F)/G_2(F) = \{P_E(F)\mu\}$.

- $\underline{\mu = P_E(F) w_2 w_1 w_3 w_4 x_{\alpha_3}(\theta) x_{\alpha_4}(\theta^\sigma)}$: In this case, $\mathrm{Stab}_G(P_E \mu) = T_{3\alpha+2\beta} \cdot \ker \Psi_E$ and we have a short exact sequence

$$\{1\} \to \ker \Psi_E \to \mathrm{Stab}_G(P_E \mu) \to T_{3\alpha+2\beta} \to \{1\}.$$

$\mathrm{H}^1(\Gamma_E, (\ker \Psi_E)(F)) = \{1\}$ since $\ker \Psi_E$ is unipotent and $\mathrm{H}^1(\Gamma_E, T_{3\alpha+2\beta}(F)) = \{1\}$ due to Hilbert 90' and thus $\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(F)}(P_E(F)\mu)\right) = \{1\}$. Hence $X_\mu(F)/G_2(F) = \{P_E(F)\mu\}$.

(3) This case is proven similarly.

$\square$

Remark 4.2. We recall that for $\mu \in H_E(F)$

$$\mathrm{Stab}_{G(F)}(P_E(F)\mu) = \mathrm{Stab}_{G(\overline{F})}\left(P_E(\overline{F})\mu\right) \cap G(F).$$

Corollary 4.2.   (1) *It holds that* $\mathrm{Stab}_{G(F)}(P_E(F)) = P(F)$.
  (2) *It holds that* $\mathrm{Stab}_{G(F)}(P_E(F) w_2 w_1) = \mathrm{Stab}_{G(\overline{F})}(P_E(F) w_2 w_3) = \mathrm{Stab}_{G(F)}(P_E(F) w_2 w_4) = L \cdot R$ *if applicable.*
  (3) *It holds that* $\mathrm{Stab}_{G(F)}(P_E(F) \mu_E) = T_{3\alpha+2\beta} \cdot \ker \Psi_E$.

5. **Unfolding of the Zeta Integral**

In this section we prove Theorem 3.2. We also denote by N_β the unipotent radical of the Borel subgroup $B \cap L$ of L.

Proof of Theorem 3.2. For $\mathfrak{Re}(s) >> 0$ it holds that

$$\frac{1}{j_E(\chi, s)} \mathcal{Z}_E(\chi, s, \varphi, f) = \int_{G(F)\backslash G(\mathbb{A})} \varphi(g)\, \mathcal{E}_E(\chi, s, f, g)\, dg$$

$$= \int_{G(F)\backslash G(\mathbb{A})} \varphi(g) \sum_{\gamma \in P_E(F)\backslash H_E(F)} f_s(\gamma g)\, dg$$

$$= \sum_{\mu \in P_E(F)\backslash H_E(F)/G(F)} I_\mu(\varphi, f_s),$$

where

$$I_\mu(\varphi, f_s) = \int_{G^\mu(F)\backslash G(\mathbb{A})} \varphi(g) f_s(\mu g)\, dg.$$

We now show that $I_\mu(\varphi, f_s) = 0$ unless μ is a representative of the open orbit.

(1) <u>If $\mu = 1$ then:</u> In this case

$$I_\mu (\varphi, f_s) = \int\limits_{P(F)\backslash G(\mathbb{A})} \varphi (g)\, dg$$

$$= \int\limits_{M(F)U(\mathbb{A})\backslash G(F)} f_s (g) \left(\int\limits_{U(F)\backslash U(\mathbb{A})} \varphi (ug)\, du \right) dg = 0,$$

since φ is cuspidal.

(2) <u>If $\mu \in \{w_2 w_1, w_2 w_3, w_2 w_4\}$ then:</u> In this case,

$$I_\mu (\varphi, f_s) = \int\limits_{L(F)\cdot R(F)\backslash G(\mathbb{A})} \varphi (g)\, f_s (\mu g)\, dg$$

$$= \int\limits_{L(F)\cdot R(\mathbb{A})\backslash G(\mathbb{A})} f_s (\mu g) \left(\int\limits_{R(F)\backslash R(\mathbb{A})} \varphi (rg)\, dr \right) dg.$$

We recall from [16, Theorem 5] that

$$\int\limits_{R(F)\backslash R(\mathbb{A})} \varphi (rg)\, dr = \sum_{\nu \in N_\beta (F)\backslash L(F)} W_\psi (\varphi) (\nu g),$$

where $W_\psi (\varphi)$ is the standard Whittaker coefficient of φ. We then have

$$I_\mu (\varphi, f_s) = \int\limits_{L(F)\cdot R(\mathbb{A})\backslash G(\mathbb{A})} f_s (\mu g) \left(\sum_{\nu \in N_\beta (F)\backslash L(F)} W_\psi (\varphi) (\nu g) \right) dg$$

$$= \int\limits_{N_\beta (\mathbb{A})\cdot R(\mathbb{A})\backslash G(\mathbb{A})} f_s (\mu g)\, W_\psi (\varphi) (\nu g) \left(\int\limits_{N_\beta (F)\backslash N_\beta (\mathbb{A})} \psi (n)\, dn \right) dg = 0.$$

(3) <u>If $\mu = \mu_E$ then:</u> Fix $\mu = \mu_E$ to be the representative of the open orbit as in Theorem 4.1. Also let $T^\mu = T_{3\alpha+2\beta}$ and $U^\mu = \ker \Psi_E$ so that $\mathrm{Stab}_G (P_E \mu) = T^\mu U^\mu$. It holds that

$$I_\mu (\varphi, f_s) = \int\limits_{T^\mu (F)U^\mu (\mathbb{A})\backslash G(\mathbb{A})} f_s (\mu g) \left(\int\limits_{U^\mu (F)\backslash U^\mu (\mathbb{A})} \varphi (ug)\, du \right) dg.$$

We now expand the function given by the inner integral along the 1-dimensional subgroup generated by the root $3\alpha+\beta$. Using cuspidality, we collapse the sum with the outer integration, and conclude that the above equals

$$I_\mu (\varphi, f_s) = \int\limits_{U^\mu (\mathbb{A})\backslash G(\mathbb{A})} f_s (\mu g) \left(\int\limits_{U(F)\backslash U(\mathbb{A})} \varphi (ug)\, \overline{\Psi_E (u)}\, du \right) dg.$$

Since $U = U^{\mu} \cdot U_{3\alpha+\beta}$ we have

$$I_{\mu}\left(\varphi, f_s\right) = \int\limits_{U(\mathbb{A})\backslash G(\mathbb{A})} \left(\int\limits_{U(F)\backslash U(\mathbb{A})} \varphi\left(ug\right) \overline{\Psi_E\left(u\right)} du\right) \left(\int\limits_{\mathbb{A}} f_s\left(\mu x_{3\alpha+\beta}\left(r\right) g\right) \psi\left(r\right) dr\right) dg.$$

$\square$

6. Unramified Computation

In this section we describe the unramified computation and recall some results from [12]. We postpone some of the more involved calculations to the appendices. For this section and the appendices we fix a place $\nu \notin S$ and drop ν from all of the notations, i.e. we assume F to be a local non-Archimedean field, E an unramified Galois étale cubic algebra over F, π an irreducible unramified representation of $G(F)$ etc. We fix on $G(F)$ the unique Haar measure λ such that $\lambda(K) = 1$, where $K = G(\mathcal{O})$.

Recall the Satake isomorphism between the spherical Hecke algebra $\mathcal{H} = \mathcal{H}(G, K)$ and the Grothendick ring $Rep\left({}^L G\right)$ described in [11]. Denote by $A_j \in \mathcal{H}$ the elements corresponding to $Sym^j\left(\mathfrak{st}\right)$ under the Satake isomorphism. In particular, for any unramified representation π and a spherical vector $v_0 \in \pi$ it holds that

$$(6.1) \qquad \int\limits_{G(F)} A_j\left(g\right) \pi\left(g\right) v_0 \, dg = \operatorname{Tr}\left(Sym^j\left(\mathfrak{st}\right)\left(t_\pi\right)\right) v_0,$$

where t_π is the Satake parameter of π.

For any such π, the Satake isomorphism induces an algebra homomorphism

$$\mathcal{H} \to \mathbb{C}$$
$$f \to \widehat{f}\left(\pi\right),$$

where $\widehat{f}\left(\pi\right)$ is given by

$$\int\limits_{G(F)} f\left(g\right) \pi\left(g\right) v_0 \, dg = \widehat{f}\left(\pi\right) v_0.$$

In particular, for any $f_1, f_2 \in \mathcal{H}$ it holds that $\widehat{f_1 * f_2} = \widehat{f_1} \cdot \widehat{f_2}$. The homomorphism $f \to \widehat{f}\left(\pi\right)$ can be extended linearly to a map of formal power series $\mathcal{H}\left[\left[T\right]\right] \to \mathbb{C}\left[\left[T\right]\right]$. Let $T = q^{-s}$.

Proposition 6.1. *For any finite order character χ of F, there exists a generating function $\Delta_{\chi,s} \in \mathcal{H}\left[\left[q^{-s}\right]\right]$, uniformly converging on a right half plane, such that for any unramified representation π with a spherical vector v_0 and **any** functional l on π, it holds that*

$$(6.2) \qquad \int\limits_{G(F)} \Delta_{\chi,s}\left(g\right) \Lambda\left(\pi\left(g\right) v_0\right) dg = \mathcal{L}\left(s, \pi, \chi, \mathfrak{st}\right) \Lambda\left(v_0\right)$$

for $\mathfrak{Re}\left(s\right) \gg 0$.

Proof. We have Poincaré's identity

$$\mathcal{L}\left(s, \pi, \chi, \mathfrak{st}\right) = \frac{1}{\det\left(\mathbb{1} - q^{-s} \mathfrak{st}\left(t_{\pi \boxtimes \chi}\right)\right)} = \prod_{i=1}^{7} \frac{1}{1 - q^{-s} \chi\left(\varpi\right) \mathfrak{st}\left(t_\pi\right)_{ii}} =$$
$$= \prod_{i=1}^{7} \left(\sum_{k=0}^{\infty} \left(q^{-s} \chi\left(\varpi\right) \mathfrak{st}\left(t_\pi\right)_{ii}\right)^k\right) = \sum_{k=0}^{\infty} \operatorname{Tr}\left(Sym^k\left(t_\pi\right)\right) \chi^k\left(\varpi\right) q^{-ks}$$

Plugging equation (6.1) in the above equality yields

$$\mathcal{L}\left(s,\pi,\chi,\mathfrak{st}\right)\Lambda\left(v_0\right) = \Lambda\left(\sum_{k=0}^{\infty}\left(\int\limits_{G(F)} A_k\left(g\right)\pi\left(g\right)v_0 dg\right)\chi^k\left(\varpi\right)q^{-ks}\right).$$

In [12] we prove that this double integral is absolutely convergent (note that χ is unitary) for $\mathfrak{Re}\left(s\right) \gg 0$ and hence one can change the order of summation and integration. The assertion then follows for

$$\Delta_{\chi,s} = \sum_{k=0}^{\infty} A_k \chi^k\left(\varpi\right)q^{-ks}.$$

$\square$

Furthermore, for any complex character Ψ of $U\left(F\right)$ and any $l \in \text{Hom}_{U(F)}\left(\pi,\mathbb{C}_\Psi\right)$ it holds that

(6.3) $$\mathcal{L}\left(s,\pi,\chi,\mathfrak{st}\right)\Lambda\left(v_0\right) = \int\limits_{U(F)\backslash G(F)}\Lambda\left(\pi\left(g\right)v_0\right)\Delta_{\chi,s}^{\Psi}\left(g\right)dg.$$

Thus, in order to prove Theorem 3.3 it is enough to prove

(6.4) $$\boxed{\Delta_{\chi,s}^{\Psi_E}\left(g\right) = F^*\left(\Psi_E,\chi,g,s\right) \quad \forall g \in G\left(F\right).}$$

We now reduce the proof of equation (6.4) to the case of $\chi = 1$. For a given unitary character χ of $F^\times$ and $s \in \mathbb{C}$ we denote

$$\chi_s\left(t\right) = \chi\left(t\right)\left|t\right|^{s+\frac{5}{2}}.$$

In appendix A.1 we prove the following result.

Proposition 6.2. *For $g = \alpha^\vee\left(t_1\right)\beta^\vee\left(t_2\right)x_\alpha\left(d\right) \in M\left(F\right)$, let*

$$\alpha_1 = \begin{cases} 1, & \left|\frac{t_2}{t_1^2}a+d\right|_{F_{\alpha_1}} \leq 1 \\ \frac{t_2}{t_1^2}a+d, & \left|\frac{t_2}{t_1^2}a+d\right|_{F_{\alpha_1}} > 1 \end{cases}, \quad \alpha_2 = \begin{cases} 1, & \left|\frac{t_2}{t_1^2}b+d\right|_{F_{\alpha_3}} \leq 1 \\ \frac{t_2}{t_1^2}b+d, & \left|\frac{t_2}{t_1^2}b+d\right|_{F_{\alpha_3}} > 1 \end{cases}, \quad \alpha_3 = \begin{cases} 1, & \left|\frac{t_2}{t_1^2}c+d\right|_{F_{\alpha_4}} \leq 1 \\ \frac{t_2}{t_1^2}c+d, & \left|\frac{t_2}{t_1^2}c+d\right|_{F_{\alpha_4}} > 1 \end{cases}.$$

Also denote $\alpha = \frac{t_1^3}{t_2}\alpha_1\alpha_2\alpha_3 \in F$. It holds that

$$F\left(\Psi_E,\chi,g,s\right) = \begin{cases} 0, & \left|\alpha\right|_F > 1 \\ \chi_s\left(\frac{t_2}{\alpha}\right)\frac{\mathcal{L}\left(s+\frac{3}{2},\chi\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)}\left(\left|\alpha\right| - \chi_s\left(\varpi\alpha\right)q\right), & \left|\frac{t_1^3}{t_2}\right|_F < 1 \end{cases}$$

It is clear that the power series of $F^*\left(\Psi_E,\chi,g,s\right)$ with respect to q^{-s} is of the form $\sum\limits_{k=0}^{\infty} B_k \chi^k\left(\varpi\right)q^{-ks}$, where the coefficients $B_k \in \mathcal{H}$ are independent of χ. Hence, in order to prove equation (6.4), it is enough to prove that

(6.5) $$\boxed{\Delta_{1,s}^{\Psi_E}\left(g\right) = F^*\left(\Psi_E,1,g,s\right) \quad \forall g \in G\left(F\right).}$$

We therefore devote the rest of this section and the appendices to prove this equality and drop χ from all notations.

Remark 6.1. Since $\Delta_s^{\Psi_E}\left(g\right), F^*\left(\Psi_E,g,s\right) \in \mathcal{M}_\Psi$, it suffices to prove the equality for $g \in M\left(F\right) \cap B\left(F\right)$.

While the right hand side of equation (6.5) is given explicitly, we do not have an explicit formula for the generating function $\Delta(g, s)$. In [12] we introduced a way to overcome this difficulty.

Recall the Cartan decomposition $G = KT^+K$, where

$$T^+ = \left\{ t \in T : |\gamma(t)| \leq 1 \ \forall \gamma \in \Phi^+ \right\}.$$

Let $D_s \in \mathcal{H}\left[\left[q^{-s}\right]\right]$ be the bi-K-invariant defined on T^+ by

$$D_s(t) = |\omega_1(t)|^{s+\frac{7}{2}} \qquad \forall t \in T^+.$$

The functions D and Δ are closely related as can be seen from the following proposition [12, Proposition 7.1 and 7.2].

Proposition 6.3. *There exists $P_s \in \mathcal{H}\left[q^{-s}\right]$ and $s_0 \in \mathbb{R}$ such that for $\mathfrak{Re}\,s > s_0$ it holds that*

$$D_s = \Delta_{s+\frac{1}{2}} * P_s \ .$$

More precisely

(6.6) $$P_s = \frac{P_0\left(q^{-s-\frac{1}{2}}\right) A_0 - P_1\left(q^{-s-\frac{1}{2}}\right) A_1}{\zeta\left(s+\frac{3}{2}\right) \zeta\left(s+\frac{7}{2}\right) \zeta\left(s+\frac{1}{2}\right)},$$

where

$$P_0(z) = \frac{z^4}{q^2} + \left(\frac{1}{q^2} + \frac{1}{q}\right) z^3 + \frac{z^2}{q} + \left(\frac{1}{q} + 1\right) z + 1, \quad P_1(z) = \frac{z^2}{q} \ .$$

Furthermore, there exists s_0 such that for any $\mathfrak{Re}(s) > s_0$ and $f \in \mathcal{M}_{\Psi_E}$, if $f * P_s \equiv 0$ then $f \equiv 0$.

Remark 6.2. Recall from [11] that $A_0 = \mathbb{1}_K$ and $A_1 = q^{-3}\left(\mathbb{1}_K + \mathbb{1}_{K\omega_1(\varpi)K}\right)$.

Remark 6.3. Since the Fourier transform is a map of $\mathcal{H}$-modules, it holds that

$$D_s^{\Psi_E} \equiv \Delta_{s+\frac{1}{2}}^{\Psi_E} * P_s$$

Corollary 6.1. *3.3 follows from*

(6.7) $$D_s^{\Psi_E} \equiv F^*(\Psi_E, \cdot, s) * P_s.$$

In order to prove equation (6.7) we compute explicitly both sides in appendix A and appendix B. The method is similar to that of the split case but while more delicate issues arise in the computation.

Remark 6.4. equation (6.7) is proved in the split case in [12] for Ψ_s as described there. It can then be deduced for $\Psi_{F \times F \times F}$ by conjugation with $m_{a,b,c}$ given in section 4. Hence, we devote appendix A and appendix B to proving equation (6.7) assuming that E is non-split.

7. Ramified Computation

In this section we prove Theorem 3.4. The proof of is similar to that of [12] and we include it for the convenience of the reader.

Proof of Theorem 3.4. Let $\mu = \mu_E$. Recall from Theorem 3.2 that

$$d_S(\chi, s, \Psi_E, \varphi_S, f_S) = \int\limits_{U_S \backslash G_S} L_{\Psi_E}(\varphi)(g) F^*(\Psi_E, \chi, g, s) \, dg = j_E(\chi, s) \int\limits_{U_S^\mu \backslash G_S} L_{\Psi_E}(\varphi)(g) f_s(\mu_E g) \, dg \ .$$

We denote by χ_s the extension of $(\chi \circ \det_{M_E}) |\det_{M_E}|^{s+\frac{5}{2}}$ to P trivial on U. For $g \in G_\mu(F_\nu)$ we have

$$\chi_s^{\mu_E}(g) = \chi_s\left(\mu_E p \mu_E^{-1}\right).$$

Since μ generates the open double coset in H_E, there is an inclusion

$$i : \operatorname{ind}_{G^{\mu_E}(F_S)}^{G(F_S)} \chi_s^{\mu_E} \hookrightarrow \operatorname{Ind}_{P_E(F_S)}^{H_E(F_S)} \chi_s$$

defined by $i(f_s)(p\mu_E g) = \chi_s(p) f_s(g)$ and $i(f_s)$ vanishes on all other double cosets $P(F_S)\mu'G(F_S)$.

For any $\phi_S \in \mathcal{S}(F_S)$ define an action of ϕ_S on π by

$$\phi_S * \varphi = \int_{F_S} \phi_S(r) \pi_S(x_{2\alpha+\beta}(r)) \varphi \, dr$$

It is easy to see that

$$L_{\Psi_E}(\pi_S(h_{3\alpha+2\beta}(t))(\phi_S * \varphi)) = \hat{\phi}_S(t) L_{\Psi_E}(\pi_S(h_{3\alpha+2\beta}(t))\varphi),$$

where $\hat{\phi}_S$ is a Fourier transform of ϕ_S.

Let us write

$$J_S(s, \varphi) = \int_{F_S^\times} L_{\Psi_E}(\varphi)(h_{3\alpha+2\beta}(t)) \mu_s(h_{3\alpha+2\beta}(t)) \, d^\times t.$$

Lemma 7.1. *For any $s_0 \in \mathbb{C}$ and any $\varphi \in \pi$ such that $L_{\Psi_E}(\varphi) \neq 0$ there exists $\phi_S \in \mathcal{S}(F_S)$ such that $J_S(s, \phi_S * \varphi) \neq 0$ around s_0.*

Proof. One has

$$\int_{F_S^\times} L_{\Psi_E}(\phi_S * \varphi)(h_{3\alpha+2\beta}(t)) \mu_s(h_{3\alpha+2\beta}(t)) \, d^\times t = \int_{F_S^\times} \hat{\phi}_S(t) \, L_{\Psi_E}(\varphi)(h_{3\alpha+2\beta}(t)) \mu_s(h_{3\alpha+2\beta}(t)) \, d^\times t.$$

Since the image of $F_S^\times$ inside F_S is locally closed we may choose ϕ_S such that $\hat{\phi}_S$ is supported on a relatively compact neighborhood of $1 \in F_S^\times$. Choose ϕ_S such that the support of $\hat{\phi}_S$ is sufficiently small to ensure the non-vanishing of $J_S(s, \phi_S * \varphi)$ around s_0. $\qquad\square$

Consider the decomposition

$$G_S = G_S^\mu \cdot T_S^c \cdot U_S^c \cdot K_S, \quad T_S^c = \{x_\beta(t) : t \in k_S^\times\}, \quad U_S^c = \{x_\alpha(r_1) x_{\alpha+\beta}(r_2) : r_i \in k_S\}$$

For any Schwarz function Φ_S on $(T^c \cdot U^c \cdot K)_S$ define $f_s(\Phi_S) \in \operatorname{ind}_{G_S^{\mu_E}}^{G_S} \chi_s^{\mu_E}$ by

$$f_s(\Phi_S)(g_\mu g) = \chi_s(g_\mu) \Phi_S(g) \quad \forall g_\mu \in G_S^\mu, \ g \in (T^c \cdot U^c \cdot K)_S.$$

Then for any $\varphi \in \pi$ it holds that

$$(7.1) \qquad\qquad d_S(\chi, s, \Psi_E, \varphi_S, f(\Phi)_s) = J_S(s, \Phi_S * \varphi)$$

By the Dixmier-Malliavin theorem [6] there exists a Schwarz function Φ_S on $(T^c \cdot U^c \cdot K)_S$ and $\varphi \in \pi$ such that $L_{\Psi_s}(\Phi_S * \varphi) \neq 0$. Then for any $\phi_S \in \mathcal{S}(k_S)$

$$d_S(\chi, s, \Psi_E, \phi_S * \varphi_S, f(\Phi_S)) = J_S(s, \phi_S * (\Phi * \varphi)).$$

By lemma 7.1 there exists a Schwarz function $\phi_S \in \mathcal{S}(k_S)$ such that $d_S(\chi, s, \Psi_E, \phi_S * \varphi_S, f_s(\Phi_S))$ is an entire function and does not vanish in a neighborhood of s_0. $\qquad\square$

8. Application: θ-lift for (G_2, S_E)

For this section let E be a Galois cubic étale algebra over F. We let χ_E be the finite order Hecke character associated to E/F by class field theory. Fix

$$n_E = \begin{cases} 2, & E = F \times F \times F \\ 1, & \text{otherwise} \end{cases}.$$

For any étale cubic algebra E over F and the choice of Chevalley-Steinberg system as in section 2 defines a splitting of the exact sequence

$$\{1\} \to H_E^{ad} \to \operatorname{Aut}(H_E) \to S_E \to \{1\}.$$

We can then form the semidirect product $H_E \rtimes S_E$. The centralizer of S_E in H_E is isomorphic to G, this gives rise to a dual reductive pair

$$G \times S_E \hookrightarrow H_E \rtimes S_E.$$

We denote the associated θ-lift by θ_{S_E}. It was shown in [8] that if a cuspidal representation π of G lies in the cuspidal image of the θ-lift from S_E then it supports the Ψ_E-Fourier coefficient and $\mathcal{L}^S(s, \pi, \chi_E, \mathfrak{st})$ admits a pole of order n_E at $s = 2$. In order to prove the converse, we recall [8, Proposition 5.1].

Proposition 8.1. *For any standard section f_s, the Eisenstein series $\mathcal{E}(\chi_E, f_s, s, g)$ has at most a pole of order n_E at $s = \frac{3}{2}$. A pole of order n_E is attained. Moreover, the space of automorphic forms*

$$Span_{\mathbb{C}}\left\{\left(s - \frac{3}{2}\right)^{n_E} \mathcal{E}_E^*(\chi_E, s, f_s, g)\Big|_{s=\frac{3}{2}}\right\},$$

is an irreducible square-integrable automorphic representation isomorphic to the minimal representation Π_E of H_E.

Theorem 8.1. *The following are equivalent*

(1) π *supports the Ψ_E-Fourier coefficient and $\mathcal{L}^S(s, \pi, \chi_E, \mathfrak{st})$ admits a pole at $s = 2$ of order n_E.*
(2) $\theta_{S_E}(\pi) \neq 0$.

Proof. As noted above, it remains to prove that (1) implies (2). Assume that $\mathcal{L}^S(s, \pi, \chi_E, \mathfrak{st})$ admits a pole of order n_E. According to Theorem 3.1, this implies that there exist f and φ such that $\mathcal{Z}_E(\chi, s, \varphi, f)$ admits a pole of order n_E. Taking the residue at equation (3.1) implies the assertion. $\square$

Remark 8.1. In fact, in an ongoing project I proved that if $\mathcal{L}(s, \pi, \chi_E, \mathfrak{st})$ admits a pole at $s = \frac{3}{2}$ of order n_E then π supports the Ψ_E-Fourier coefficient and so one may drop this from the assumption in (1).

Appendices

Appendix A. Computing $F^*(\Psi_E, \chi, \cdot, s) * P_s(\cdot)$

A.1. **Computing $F^*(\Psi_E, \chi, \cdot, s)$.** In this subsection we prove Proposition 6.2.

Remark A.1. For a triple $r = (r_1, r_3, r_4)$ we write $x_\alpha(r) = x_{\alpha_1}(r_1) x_{\alpha_3}(r_3) x_{\alpha_4}(r_4)$.

Proof of Proposition 6.2. Using left (P_E, χ_s)-invariance and right K-invariance we have

$$F^*\left(\Psi_E, \chi, g, s\right) = \int_F f_s\left(w_\beta w_\alpha x_\alpha\left(e\right) x_{3\alpha+\beta}\left(r\right) \alpha^\vee\left(t_1\right) \beta^\vee\left(t_2\right) x_\alpha\left(d\right)\right) \psi\left(r\right) dr =$$

$$= \chi_s\left(\frac{t_2^2}{t_1^3}\right) \int_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\right) x_{-\alpha}\left(\frac{t_2}{t_1^2}e + d\right)\right) \psi\left(r\right) dr.$$

We continue to consider this integral case-by-case.

Case I: Assume that $\left|\frac{t_2}{t_1^2}a + d\right|_{F_{\alpha_1}}, \left|\frac{t_2}{t_1^2}b + d\right|_{F_{\alpha_3}}, \left|\frac{t_2}{t_1^2}c + d\right|_{F_{\alpha_4}} \leq 1$.

In this case we have

$$F^*\left(\Psi_E, \chi, g, s\right) = \chi_s\left(\frac{t_2^2}{t_1^3}\right) \int_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\right)\right) \psi\left(r\right) dr.$$

Denote $\alpha = \frac{t_1^3}{t_2}$. We split the integral in two

$$\int_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\right)\right) \psi\left(r\right) dr = \int_{\alpha\mathcal{O}} f_s\left(w_\beta x_\beta\left(\frac{r}{\alpha}\right)\right) \psi\left(r\right) dr + \int_{F\backslash\alpha\mathcal{O}} f_s\left(w_\beta x_\beta\left(\frac{r}{\alpha}\right)\right) \psi\left(r\right) dr.$$

Considering the first integral yields

$$\int_{\alpha\mathcal{O}} f_s\left(w_\beta x_\beta\left(\frac{r}{\alpha}\right)\right) \psi\left(r\right) dr = \int_{\alpha\mathcal{O}} \psi\left(r\right) dr = \begin{cases} \left|\alpha\right|_F, & \left|\alpha\right|_F \leq 1 \\ 0, & \left|\alpha\right|_F > 1 \end{cases},$$

while the second integral equals

$$\int_{F\backslash\alpha\mathcal{O}} f_s\left(w_\beta x_\beta\left(\frac{r}{\alpha}\right)\right) \psi\left(r\right) dr = \int_{F\backslash\alpha\mathcal{O}} f_s\left(x_{-\beta}\left(\frac{r}{\alpha}\right)\right) \psi\left(r\right) dr =$$

$$= \int_{F\backslash\alpha\mathcal{O}} f_s\left(x_\beta\left(\frac{\alpha}{r}\right) \beta^\vee\left(\frac{\alpha}{r}\right)\right) \psi\left(r\right) dr =$$

$$= \chi_s\left(\alpha\right) \int_{F\backslash\alpha\mathcal{O}} \chi_s\left(\frac{1}{r}\right) \psi\left(r\right) dr.$$

Since χ_s is $\mathcal{O}^\times$-invariant this equals

$$\int_{F\backslash\alpha\mathcal{O}} \chi_s\left(\frac{1}{r}\right) \psi\left(r\right) dr = \sum_{j=1-val(\alpha)}^{\infty} \chi_s\left(\varpi^j\right) \int_{|r|=q^j} \psi\left(r\right) dr.$$

Recall that

$$\int_{|r|=q^j} \psi\left(r\right) dr = \begin{cases} 0, & j > 1 \\ -1, & j = 1 \\ q^j\left(1 - q^{-1}\right), & j \leq 0 \end{cases}.$$

Inserting this to the previous expression yields

$$\int_{F\backslash\alpha\mathcal{O}} \chi_s\left(\frac{1}{r}\right)\psi\left(r\right)dr = \begin{cases} 0, & |\alpha|_F > 1 \\ -\chi_s\left(\varpi\right), & |\alpha|_F = 1 \\ \left(1-q^{-1}\right)\left(\frac{1-\chi_s\left(\frac{1}{\alpha}\right)|\alpha|_F}{1-\chi_s\left(\varpi^{-1}\right)|\varpi|_F}\right) - \chi_s\left(\varpi\right), & |\alpha|_F < 1 \end{cases}.$$

Combining all of the above yields

$$F^*\left(\Psi_E, \chi, g, s\right) = \begin{cases} 0, & |\alpha|_F > 1 \\ \chi_s\left(t_2\right)\left(1-\chi_s\left(\varpi\right)\right), & |\alpha|_F = 1 \\ \chi_s\left(\frac{t_2}{\alpha}\right)\left(|\alpha|_F + \chi_s\left(\alpha\right)\left(\left(1-q^{-1}\right)\frac{1-\chi_s\left(\frac{1}{\alpha}\right)|\alpha|_F}{1-\chi_s\left(\varpi^{-1}\right)|\varpi|_F} - \chi_s\left(\varpi\right)\right)\right), & \left|\frac{t_1^3}{t_2}\right|_F < 1 \end{cases}.$$

Note that

$$\chi_s\left(\frac{t_2}{\alpha}\right)\left(|\alpha|_F + \chi_s\left(\alpha\right)\left(\left(1-q^{-1}\right)\frac{1-\chi_s\left(\frac{1}{\alpha}\right)|\alpha|_F}{1-\frac{1}{\chi_s\left(\varpi\right)q}} - \chi_s\left(\varpi\right)\right)\right) = \chi_s\left(\frac{t_2}{\alpha}\right)\frac{\mathcal{L}\left(s+\frac{3}{2},\chi\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)}\left(|\alpha|_F - \chi_s\left(\varpi\,\alpha\right)q\right)$$

and hence we may conclude that

$$F^*\left(\Psi_E, \chi, g, s\right) = \begin{cases} 0, & |\alpha|_F > 1 \\ \frac{\chi_s\left(t_2\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)}, & |\alpha|_F = 1 \\ \chi_s\left(\frac{t_2}{\alpha}\right)\frac{\mathcal{L}\left(s+\frac{3}{2},\chi\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)}\left(|\alpha|_F - \chi_s\left(\varpi\,\alpha\right)q\right), & \left|\frac{t_1^3}{t_2}\right|_F < 1 \end{cases}.$$

Case II: The case where $\left|\frac{t_2}{t_1^2}a + d\right|_{F_{\alpha_1}} > 1$, $\left|\frac{t_2}{t_1^2}b + d\right|_{F_{\alpha_3}} \leq 1$ and $\left|\frac{t_2}{t_1^2}c + d\right|_{F_{\alpha_4}} \leq 1$ (or the cases equivalent to it) can not happen under **(CT)** for E non-split.

Case III: Assume $\left|\frac{t_2}{t_1^2}a + d\right|_{F_{\alpha_1}} \leq 1$, $\left|\frac{t_2}{t_1^2}b + d\right|_{F_{\alpha_3}} > 1$ and $\left|\frac{t_2}{t_1^2}c + d\right|_{F_{\alpha_4}} > 1$ (the cases equivalent to it follow similarly). We then have

$$F^*\left(\Psi_E, \chi, g, s\right) = \chi_s\left(\frac{t_2^2}{t_1^3}\right)\int_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\right)x_{-\alpha_1}\left(\frac{t_2}{t_1^2}b + d\right)x_{-\alpha_3}\left(\frac{t_2}{t_1^2}c + d\right)\right)\psi\left(r\right)dr =$$

$$= \chi_s\left(\frac{t_2^2}{t_1^3}\right)\int_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\left(\frac{1}{\frac{t_2}{t_1^2}b + d}\cdot\frac{1}{\frac{t_2}{t_1^2}c + d}\right)\right)\right)\psi\left(r\right)dr$$

Denote $\alpha = \frac{t_1^3}{t_2}\left(\frac{t_2}{t_1^2}b + d\right)\left(\frac{t_2}{t_1^2}c + d\right)$, we then have as before

$$\int_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\left(\frac{1}{\frac{t_2}{t_1^2}b + d}\cdot\frac{1}{\frac{t_2}{t_1^2}c + d}\right)\right)\right)\psi\left(r\right)dr = \int_{\alpha\mathcal{O}}\psi\left(r\right)dr + \chi_s\left(\alpha\right)\int_{F\backslash\alpha\mathcal{O}}\chi_s\left(\frac{1}{r}\right)\psi\left(r\right)dr.$$

Repeating the evaluation of the first case yields the assertion.

Case IV: Assume $\left|\frac{t_2}{t_1^2}a+d\right|_{F_{\alpha_1}} > 1$, $\left|\frac{t_2}{t_1^2}b+d\right|_{F_{\alpha_3}} > 1$ and $\left|\frac{t_2}{t_1^2}c+d\right|_{F_{\alpha_4}} > 1$. We then have

$$F^*\left(\Psi_E, \chi, g, s\right) = \chi_s\left(\frac{t_2^2}{t_1^3}\right) \int_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\right) x_{-\alpha}\left(\frac{t_2}{t_1^2}e+d\right)\right)\psi(r)\,dr =$$

$$= \chi_s\left(\frac{\frac{t_2^2}{t_1^3}}{\mathrm{Nm}\left(\frac{1}{\frac{t_2}{t_1^2}e+d}\right)}\right) \int_F f_s\left(w_\beta x_\beta\left(\frac{\frac{t_2}{t_1^3}r}{\mathrm{Nm}\left(\frac{t_2}{t_1^2}e+d\right)}\right) x_\alpha\left(\frac{t_2}{t_1^2}e+d\right)\right)\psi(r)\,dr,$$

Denote $\alpha = \frac{t_1^3}{t_2}\left(\frac{t_2}{t_1^2}a+d\right)\left(\frac{t_2}{t_1^2}b+d\right)\left(\frac{t_2}{t_1^2}c+d\right)$. It holds that

$$\int_F f_s\left(w_\beta x_\beta\left(\frac{r}{\alpha}\right) x_\alpha\left(\frac{t_2}{t_1^2}e+d\right)\right)\psi(r)\,dr = \int_{\alpha\mathcal{O}}\psi(r)\,dr + \chi_s(\alpha)\int_{F\setminus\alpha\mathcal{O}}\chi_s\left(\frac{1}{r}\right)\psi(r)\,dr.$$

Repeating the evaluation of the first case yield the assertion once again.

$\square$

As explained in section 6, in what follows we may assume that $\chi = \mathbb{1}$. We now obtain a more explicit formula for $F^*\left(\Psi_E, \mathbb{1}, g, s\right)$.

Corollary A.1. *Let $g = h_\alpha(t_1)h_\beta(t_2)x_\alpha(d)$.*

- $\underline{E = F \times K}$*: For $d = 0$ it holds that*

$$F^*\left(\Psi_E, \mathbb{1}, g, s\right) = \begin{cases} \frac{\zeta(5s-1)}{\zeta(5s)}|t_2|_F^{10s-1}|t_1|_F^{3-15s}\left(1-\left|\frac{t_1^3}{t_2}\right|_F^{5s-1}q^{1-5s}\right), & \left|\frac{t_2}{t_1^2}\right|_F \le 1 \\ \frac{\zeta(5s-1)}{\zeta(5s)}|t_2|_F|t_1|_F^{5s-1}\left(1-\left|\frac{t_2}{t_1}\right|_F^{5s-1}q^{1-5s}\right), & \left|\frac{t_2}{t_1^2}\right|_F > 1 \end{cases}$$

and for $d \notin \mathcal{O}$, it holds that

$$F^*\left(\Psi_E, \mathbb{1}, g, s\right) = \begin{cases} \frac{\zeta(5s-1)}{\zeta(5s)}|t_2|_F\left|\frac{d}{t_1}\right|_F^{1-5s}\left(1-\left|\frac{dt_2}{t_1}\right|_F^{5s-1}q^{1-5s}\right), & \left|\frac{dt_1^2}{t_2}\right|_F \le 1\,\&\,\left|\frac{dt_2}{t_1}\right|_F \le 1 \\ \frac{\zeta(5s-1)}{\zeta(5s)}|t_2|_F^{10s-1}|dt_1|_F^{3-15s}\left(1-\left|\frac{d^3t_1^3}{t_2}\right|_F^{5s-1}q^{1-5s}\right), & \left|\frac{dt_1^2}{t_2}\right|_F > 1\,\&\,\left|\frac{d^3t_1^3}{t_2}\right|_F \le 1 \\ 0, & \left|\frac{dt_1^2}{t_2}\right|_F \le 1\,\&\,\left|\frac{dt_2}{t_1}\right|_F > 1 \\ & \left|\frac{dt_1^2}{t_2}\right|_F > 1\,\&\,\left|\frac{d^3t_1^3}{t_2}\right|_F > 1 \end{cases}.$$

- $\underline{E \text{ a field}}$*: For $d = 0$ it holds that*

$$F^*\left(\Psi_E, \mathbb{1}, g, s\right) = \begin{cases} \frac{\zeta(5s-1)}{\zeta(5s)}|t_2|_F^{10s-1}|t_1|_F^{3-15s}\left(1-\left|\frac{t_1^3}{t_2}\right|_F^{5s-1}q^{1-5s}\right), & \left|\frac{t_2}{t_1^2}\right|_F \le 1 \\ \frac{\zeta(5s-1)}{\zeta(5s)}|t_2|_F^{2-5s}|t_1|_F^{15s-3}\left(1-\left|\frac{t_2^2}{t_1^3}\right|_F^{5s-1}q^{1-5s}\right), & \left|\frac{t_2}{t_1^2}\right|_F > 1 \end{cases}$$

and for $d \notin \mathcal{O}$, it holds that

$$F^* \left(\Psi_E, \mathbb{1}, g, s\right) = \begin{cases} \frac{\zeta(5s-1)}{\zeta(5s)} |t_2|_F^{2-5s} |t_1|_F^{15s-3} \left(1 - \left|\frac{t_2^2}{t_1^3}\right|_F^{5s-1} q^{1-5s}\right), & \left|\frac{dt_1^2}{t_2}\right|_F \leq 1 \& \left|\frac{t_2^2}{t_1^3}\right|_F \leq 1 \\ \frac{\zeta(5s-1)}{\zeta(5s)} |t_2|_F^{10s-1} |dt_1|_F^{3-15s} \left(1 - \left|\frac{d^3 t_1^3}{t_2}\right|_F^{5s-1} q^{1-5s}\right), & \left|\frac{dt_1^2}{t_2}\right|_F > 1 \& \left|\frac{d^3 t_1^3}{t_2}\right|_F \leq 1 \\ 0, & \left|\frac{dt_1^2}{t_2}\right|_F \leq 1 \& \left|\frac{t_2^2}{t_1^3}\right|_F > 1 \\ & \left|\frac{dt_1^2}{t_2}\right|_F > 1 \& \left|\frac{d^3 t_1^3}{t_2}\right|_F > 1 \end{cases}.$$

A.2. Computing $F^* \left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s \left(\cdot\right)$.

We recall from section 6 that

$$P_s = \frac{P_0 \left(q^{-s-\frac{1}{2}}\right) A_0 - P_1 \left(q^{-s-\frac{1}{2}}\right) A_1}{\zeta\left(s+\frac{3}{2}\right) \zeta\left(s+\frac{7}{2}\right) \zeta\left(s+\frac{1}{2}\right)}.$$

As $A_0 = \mathbb{1}_K$ and $A_1 = q^{-3} \left(\mathbb{1}_K + \mathbb{1}_{K\omega_1(\varpi)K}\right)$, in order to compute $F^* \left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s \left(\cdot\right)$ we need only to compute $F^* \left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_1(\varpi)K}$. For any $f \in \mathcal{M}_{\Psi_E}$ it holds that

$$f * \mathbb{1}_{K\omega_1(\varpi)K} \left(g\right) = \int_{K\omega_1(\varpi)K} f\left(gh\right) dh = \sum_{\gamma \in (K\omega_1(\varpi)K)/K} f\left(g\gamma\right).$$

Let $g = h_\alpha \left(t_1\right) h_\beta \left(t_2\right) x_\alpha \left(d\right)$ and assume that if $d \in \mathcal{O}$ then $d = 0$ (we may do this do to right K-invariance). Fixing the list of representatives of $\left(K\omega_1\left(\varpi\right) K\right)/K$ obtained in [12, Appendix A.3] we obtain

(A.1)

$$\begin{aligned} \left(F^* \left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_1(\varpi)K}\right) \left(g\right) =& F_s \left(\left(\frac{t_1}{\varpi}, \frac{t_2}{\varpi^2}\right) x_\alpha \left(d\right)\right) + q^6 F_s \left(\left(\varpi t_1, \varpi^2 t_2\right) x_\alpha \left(d\right)\right) + \\ & q F_s \left(\left(\frac{t_1}{\varpi}, \frac{t_2}{\varpi}\right) x_\alpha \left(\varpi d\right)\right) + q \sum_{s \in \mathcal{O}/(\varpi)} F_s \left(\left(t_1, \frac{t_2}{\varpi}\right) x_\alpha \left(\frac{d-s}{\varpi}\right)\right) + \\ & q^4 F_s \left(\left(t_1, \varpi t_2\right) x_\alpha \left(\varpi d\right)\right) + q^4 \sum_{s \in \mathcal{O}/(\varpi)} F_s \left(\left(\varpi t_1, \varpi t_2\right) x_\alpha \left(\frac{d-s}{\varpi}\right)\right) + \\ & \left(GS \left(\Psi_E, g\right)\right) F_s \left(\left(t_1, t_2\right) x_\alpha \left(d\right)\right), \end{aligned}$$

where GS denotes the following *Gaussian sum*:
(A.2)

$$GS \left(\Psi_E, g\right) = \left(q-1\right) + q \sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E \left(g x_\beta \left(\frac{r}{\varpi}\right) g^{-1}\right) + q \sum_{\substack{r, y \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E \left(g u \left(\frac{y^3 r}{\varpi}, \frac{y^2 r}{\varpi}, \frac{yr}{\varpi}, \frac{r}{\varpi}\right)^{-1} g^{-1}\right).$$

Lemma A.1. • $\underline{E = F \times K}$*: For $d = 0$ it holds that*

$$GS \left(\Psi_E, g\right) = \begin{cases} q^3 - 1, & \left|\frac{t_2}{t_1^2}\right| \leq 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| < 1 \ or \ \left|\frac{t_2}{t_1^2}\right| > 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| < 1 \\ -1, & \left|\frac{t_2}{t_1^2}\right| \leq 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| = 1 \\ q^2 - 1. & \left|\frac{t_2}{t_1^2}\right| > 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| = 1 \end{cases}$$

and for $d \notin \mathcal{O}$, it holds that

$$GS\left(\Psi_E, g\right) = \begin{cases} q^3 - 1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \ \& \ \left|\frac{dt_2}{t_1}\right| < 1 \ or \ \left|\frac{dt_1^2}{t_2}\right| > 1 \ \& \ \left|\frac{d^3 t_1^3}{t_2}\right| < 1 \\ -1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \ \& \ \left|\frac{dt_2}{t_1}\right| = 1 \ or \ \left|\frac{dt_1^2}{t_2}\right| > \ 1 \& \ \left|\frac{d^3 t_1^3}{t_2}\right| = 1 \\ q - 1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \ \& \ \left|\frac{dt_2}{t_1}\right| > 1 \ or \ \left|\frac{dt_1^2}{t_2}\right| > 1 \ \& \ \left|\frac{d^3 t_1^3}{t_2}\right| > 1 \end{cases}.$$

- *E a field: For $d = 0$ it holds that*

$$GS\left(\Psi_E, g\right) = \begin{cases} q^3 - 1, & \left|\frac{t_2}{t_1^2}\right| \leq 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| < 1 \ or \ \left|\frac{t_2}{t_1^2}\right| > 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| < 1 \\ -1, & \left|\frac{t_2}{t_1^2}\right| < 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| = 1 \ or \ \left|\frac{t_2}{t_1^2}\right| > 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| = 1 \\ -q^2 - 1, & \left|\frac{t_2}{t_1^2}\right| = 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| = 1 \end{cases}$$

and for $d \notin \mathcal{O}$, it holds that

$$GS\left(\Psi_E, g\right) = \begin{cases} q^3 - 1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| < 1 \ or \ \left|\frac{dt_1^2}{t_2}\right| > 1 \ \& \ \left|\frac{d^3 t_1^3}{t_2}\right| < 1 \\ -1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| = 1 \ or \ \left|\frac{dt_1^2}{t_2}\right| > 1 \ \& \ \left|\frac{d^3 t_1^3}{t_2}\right| = 1 \\ q - 1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| > 1 \ or \ \left|\frac{dt_1^2}{t_2}\right| > 1 \ \& \ \left|\frac{d^3 t_1^3}{t_2}\right| > 1 \end{cases}.$$

Proof. We prove this lemma only for the case where $d = 0$, the case $d \notin \mathcal{O}$ is proven in a similar way. The proof relies on the fact that

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi\left(\varpi^n x\right) = \begin{cases} 0, & n < -1 \\ -1, & n = -1 \\ q - 1, & n \geq 0 \end{cases}$$

We first deal with the first summand in equation (A.2)

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E\left(g x_\beta\left(\frac{r}{\varpi}\right) g^{-1}\right) = \sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi\left(N_{a,b,c} \frac{t_2^2}{t_1^3} \frac{r_1}{\varpi}\right).$$

- $\underline{E = F \times K:}$ In this case $N_{a,b,c} = 0$ due to **(CT)** and hence

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E\left(g x_\beta\left(\frac{r}{\varpi}\right) g^{-1}\right) = q - 1.$$

- $\underline{E \text{ a field:}}$ In this case $N_{a,b,c} \in \mathcal{O}^\times$ and hence

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E\left(g x_\beta\left(\frac{r}{\varpi}\right) g^{-1}\right) = \begin{cases} q - 1, & \left|\frac{t_2^2}{t_1^3}\right| < 1 \\ -1, & \left|\frac{t_2^2}{t_1^3}\right| = 1 \end{cases}.$$

Here we assumed that $\left|\frac{t_2^2}{t_1^3}\right| \leq 1$ in accordance with Remark B.2.

We now compute the second summand in equation (A.2)

$$\sum_{\substack{r,y \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E \left(gu \left(\frac{y^3 r}{\varpi}, \frac{y^2 r}{\varpi}, \frac{yr}{\varpi}, \frac{r}{\varpi} \right)^{-1} g^{-1} \right) =$$

$$\sum_{\substack{r,y \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi \left(\frac{r_1}{\varpi} \left(N_E \frac{t_2^2}{t_1^3} y^3 + D_E \frac{t_2}{t_1} y^2 + \frac{t_1^3}{t_2} \right) \right) =$$

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi \left(\frac{r_1}{\varpi} \frac{t_1^3}{t_2} \right) + \sum_{\substack{y \in \mathcal{O}/(\varpi) \\ y \not\equiv 0}} \sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi \left(\frac{r_1}{\varpi} \left(N_E \frac{t_2^2}{t_1^3} y^3 + D_E \frac{t_2}{t_1} y^2 + \frac{t_1^3}{t_2} \right) \right).$$

We first have

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi \left(\frac{r_1}{\varpi} \frac{t_1^3}{t_2} \right) = \begin{cases} q-1, & \left| \frac{t_1^3}{t_2} \right| < 1 \\ -1, & \left| \frac{t_1^3}{t_2} \right| = 1 \end{cases},$$

Where we assumed that $\left| \frac{t_2^2}{t_1^3} \right| \leq 1$ in accordance with Remark B.2.

- $\underline{E = F \times K}$: In this case $N_{a,b,c} = 0$. For $y \not\equiv 0$ it then holds that

$$\left| \frac{r_1}{\varpi} \left(N_E \frac{t_2^2}{t_1^3} y^3 + D_E \frac{t_2}{t_1} y^2 + \frac{t_1^3}{t_2} \right) \right| = \left| \frac{t_1^3}{t_2} \left(D_E \left(\frac{t_2}{t_1} y \right)^2 + 1 \right) \right| = \begin{cases} \left| \frac{t_1^3}{t_2} \right|, & \left| \frac{t_2^2}{t_1^2} \right| \leq 1 \\ \left| \frac{t_2}{t_1} \right|, & \left| \frac{t_2^2}{t_1^2} \right| > 1 \end{cases}$$

and hence

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi \left(\frac{r_1}{\varpi} \left(N_E \frac{t_2^2}{t_1^3} y^3 + D_E \frac{t_2}{t_1} y^2 + \frac{t_1^3}{t_2} \right) \right) = \begin{cases} q-1, & \left| \frac{t_2}{t_1^2} \right| \leq 1 \& \left| \frac{t_1^3}{t_2} \right| < 1 \\ & \left| \frac{t_2}{t_1^2} \right| > 1 \& \left| \frac{t_2}{t_1} \right| < 1 \\ -1, & \left| \frac{t_2}{t_1^2} \right| \leq 1 \& \left| \frac{t_1^3}{t_2} \right| = 1 \\ & \left| \frac{t_2}{t_1^2} \right| > 1 \& \left| \frac{t_2}{t_1} \right| = 1 \end{cases},$$

- $\underline{E \text{ a field}}$: We further assume that $\left| \frac{t_1^3}{t_2} \right| \leq \left| \frac{t_2^2}{t_1^3} \right| \leq 1$ in accordance with Remark B.2. In this case $N_{a,b,c} \in \mathcal{O}$. For $y \not\equiv 0$ it then holds that

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi \left(\frac{r_1}{\varpi} \left(N_E \frac{t_2^2}{t_1^3} y^3 + D_E \frac{t_2}{t_1} y^2 + \frac{t_1^3}{t_2} \right) \right) = \left| \frac{t_1^3}{t_2} \left(N_E \left(\frac{t_2}{t_1^2} y \right)^3 + D_E \left(\frac{t_2}{t_1^2} y \right)^2 + 1 \right) \right| = \begin{cases} \left| \frac{t_1^3}{t_2} \right|, & \left| \frac{t_2}{t_1^2} \right| \leq 1 \\ \left| \frac{t_2^3}{t_1^3} \right|, & \left| \frac{t_2}{t_1^2} \right| > 1 \end{cases}$$

and hence

$$\sum_{\substack{r_1 \in A \\ r_1 \not\equiv 0}} \psi \left(\frac{t_1^3}{t_2} \frac{r_1}{\varpi} \left(N_E \left(\frac{t_2}{t_1^2} y \right)^3 + D_E \left(\frac{t_2}{t_1^2} y \right)^2 + 1 \right) \right) = \begin{cases} q-1, & \left| \frac{t_2}{t_1^2} \right| \leq 1 \& \left| \frac{t_1^3}{t_2} \right| < 1 \\ & \left| \frac{t_2}{t_1^2} \right| > 1 \& \left| \frac{t_2^2}{t_1^3} \right| < 1 \\ -1, & \left| \frac{t_2}{t_1^2} \right| \leq 1 \& \left| \frac{t_1^3}{t_2} \right| = 1 \\ & \left| \frac{t_2}{t_1^2} \right| > 1 \& \left| \frac{t_2^2}{t_1^3} \right| = 1 \end{cases},$$

Combining all of the above yields the assertion.

$\square$

We now turn to compute $F^* \left(\Psi_E, \mathbb{1}, \cdot, s \right) * P_s \left(\cdot \right) \left(g \right)$. As this is computation boils down to inserting Corollary A.1 and equation (A.2) into equation (A.1) we compute this only for $g = 1$.

Denote $P = P_0 \left(q^{-s-\frac{1}{2}} \right)$ and $Q = P_1 \left(q^{-s-\frac{1}{2}} \right)$.

- $\underline{E = F \times K}$: In this case

$$\left(F^* \left(\Psi_E, \cdot, s \right) * P_s \left(\cdot \right) \right) \left(1 \right) =$$

$$\frac{\zeta_F \left(s + \frac{5}{2} \right) \zeta_K \left(s + \frac{3}{2} \right) \zeta_F \left(2s + 1 \right)}{\zeta_F \left(s + \frac{3}{2} \right) \zeta_F \left(s + \frac{7}{2} \right) \zeta_F \left(s + \frac{1}{2} \right)} \left[\frac{P - q^{-3} Q}{\zeta_F \left(s + \frac{5}{2} \right)} - \frac{q^{-3} Q}{\zeta_F \left(s + \frac{5}{2} \right)} \left(\frac{q^{-s-\frac{5}{2}} \zeta_F \left(s + \frac{3}{2} \right)}{\zeta_F \left(2s - 1 \right)} + q^{-\left(s - \frac{3}{2} \right)} - 1 \right) \right] = \frac{1}{\zeta_F \left(s + \frac{7}{2} \right)}.$$

- $\underline{E \text{ cubic Galois field extension of } F}$: In this case

$$\left(F^* \left(\Psi_E, \cdot, s \right) * P_s \left(\cdot \right) \right) \left(1 \right) =$$

$$\frac{\zeta_F \left(s + \frac{5}{2} \right) \zeta_E \left(s + \frac{3}{2} \right) \zeta_F \left(2s + 1 \right)}{\zeta_F \left(s + \frac{3}{2} \right)^2 \zeta_F \left(s + \frac{7}{2} \right) \zeta_F \left(s + \frac{1}{2} \right)} \left[\frac{P - q^{-3} Q}{\zeta_F \left(s + \frac{5}{2} \right)} - \frac{q^{-3} Q}{\zeta_F \left(s + \frac{5}{2} \right)} \left(\frac{q^{-s-\frac{5}{2}} \zeta_F \left(s + \frac{3}{2} \right)}{\zeta_F \left(2s - 1 \right)} - q2 - 1 \right) \right] = \frac{1}{\zeta_F \left(s + \frac{3}{2} \right) \zeta_F \left(s + \frac{7}{2} \right)}.$$

Appendix B. Computation of $D_s^{\Psi_E}$

In this section we compute the Ψ_E-Fourier coefficient of D_s. Since Theorem 3.3 was already proved when $E = F \times F \times F$, we restrict ourselves to the assumption that E is a non-split Galois étale cubic algebra over F.

Theorem B.1. $D_s^{\Psi_E} \left(g \right) = 0$ unless $g \in UTK$. If $g = h_\alpha \left(t_1 \right) h_\beta \left(t_2 \right)$ it holds that

- If $N_{a,b,c} = 0$ then

$$D_s^{\Psi_E} \left(g \right) = \begin{cases} \dfrac{q^{-\left(s + \frac{7}{2} \right) n}}{\zeta_F \left(s + \frac{7}{2} \right)}, & \left| \dfrac{t_1^2}{t_2} \right| = 1 \\ 0, & \left| \dfrac{t_1^2}{t_2} \right| > 1 \\ \dfrac{q^{2n - m - \left(s + \frac{7}{2} \right) n}}{\zeta_F \left(s + \frac{3}{2} \right) \zeta_F \left(s + \frac{7}{2} \right)}, & \left| \dfrac{t_1^2}{t_2} \right| < 1 \end{cases}$$

- If $N_{a,b,c} \in \mathcal{O}^\times$ then

$$D_s^{\Psi_E} \left(g \right) = \begin{cases} \dfrac{q^{-\left(s + \frac{7}{2} \right) n}}{\zeta_F \left(s + \frac{3}{2} \right) \zeta_F \left(s + \frac{7}{2} \right)}, & \left| \dfrac{t_1^2}{t_2} \right| = 1 \\ 0, & \left| \dfrac{t_1^2}{t_2} \right| > 1 \\ 0, & \left| \dfrac{t_1^2}{t_2} \right| < 1 \end{cases}$$

Before performing a direct computation of $D_s^{\Psi_E} \left(g \right)$ we make some preparations. We let SO_7 be the special orthogonal group that preserves the split symmetric form $\left(\delta_{i,7-i} \right)$ viewed as a subgroup of GL_7. We fix an embedding $\iota : SO_7 \left(F \right) \hookrightarrow G \left(F \right)$ as in [14]. In particular,

For $h = u(r_1, r_2, r_3, r_4, r_5) h_\alpha(t_1) h_\beta(t_2) x_\alpha(d)$ it holds that

(B.1)

$$\iota(h) = \begin{pmatrix} 1 & 0 & r_2 & r_3 & \frac{-r_4}{2} & \frac{r_2 r_3 + r_5}{2} & \frac{r_2 r_4 - r_3^2}{2} \\ 0 & 1 & r_1 & r_2 & \frac{-r_3}{2} & \frac{r_1 r_3 - r_2^2}{2} & \frac{r_1 r_4 - 2 r_2 r_3 - r_5}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{r_3}{2} & \frac{r_4}{2} \\ 0 & 0 & 0 & 1 & 0 & -r_2 & -r_3 \\ 0 & 0 & 0 & 0 & 1 & -r_1 & -r_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t_1 & & & & & & \\ & \frac{t_2}{t_1} & & & & & \\ & & \frac{t_1^2}{t_2} & & & & \\ & & & 1 & & & \\ & & & & \frac{t_2}{t_1^2} & & \\ & & & & & \frac{t_1}{t_2} & \\ & & & & & & \frac{1}{t_1} \end{pmatrix} \cdot \begin{pmatrix} 1 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d & -\frac{d^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following results are simple to check and will be useful in what follows.

Lemma B.1. *The function* $\Gamma : G(F) \to \mathbb{R}$ *given by*

$$\Gamma(g) = \max_{1 \le i,j \le 7} |\iota(g)_{i,j}|$$

is a bi-K-invariant function and for $t \in T^+$ *it satisfies*

$$\Gamma(t) = |\omega_1(t)|^{-1}.$$

Thus, we may write $D(g,s) = \sum_{k=0}^{\infty} D_k(g) q^{-(s+\frac{7}{2})k}$, where

$$D_k(g) = \begin{cases} 1, & \Gamma(g) = q^k \\ 0, & \text{otherwise} \end{cases}.$$

For any $g \in G$ define $U_k(g) = \{u \in U : \Gamma(ug) \le q^k\}$ and let

$$E_k(g) = \begin{cases} 1, & \Gamma(g) \le q^k \\ 0, & \text{otherwise} \end{cases}.$$

Obviously

$$D_k(g) = E_k(g) - E_{k-1}(g)$$

and in particular

$$D_s^{\Psi_E}(g,s) = \sum_{k=0}^{\infty} \left(E_k^{\Psi_E}(g) - E_{k-1}^{\Psi_E}(g) \right) q^{-(s+\frac{7}{2})k}.$$

Hence, in order to compute $D_s^{\Psi_E}(g)$ we compute

$$E_k^{\Psi_E}(g) = \int_{U_k(g)} \Psi_E(u) \, du.$$

Lemma B.2. *For* $n_1, n_2, n_3 \in \mathbb{N}$ *with* $n_1 + n_2 \ge n_3$ *let*

$$\kappa(n_1, n_2, n_3) := \lambda \left\{ (x,y) \in F^2 : |x| \le q^{n_1}, \quad |y| \le q^{n_2}, \quad |xy| \le q^{n_3} \right\}$$

it holds that

$$\kappa(n_1, n_2, n_3) := q^{n_3} \left(1 + (n_1 + n_2 - n_3) \left(1 - q^{-1} \right) \right),$$

where λ *is the Haar measure on* G *such that* $\lambda(K) = 1$.

For $n_1, n_2 \in \mathbb{N}$ let

$$\kappa^{(n_1)}_{(a,b,c)} = \lambda \left\{ (x,y) \in F^2 : |x|, |y|, \left| P_{(a,b,c)}\left(\frac{x}{y}\right) \right| \le q^{n_1} \right\}$$

$$\kappa^{(n_1)}_{(a,b,c)}(q^{n_2}) = \lambda \left\{ (x,y) \in F^2 : |x| = |y| = q^{n_2}, \left| P_{(a,b,c)}\left(\frac{x}{y}\right) \right| \le q^{n_1 - n_2 \deg\left(P_{(a,b,c)}\right)} \right\},$$

where

$$P_{(a,b,c)}(z) = \begin{cases} D_{(a,b,c)} + z^2, & N_{(a,b,c)=0} \\ N_{(a,b,c)} + D_{(a,b,c)}z + z^3, & N_{(a,b,c)} \in \mathcal{O}^\times \end{cases}.$$

Lemma B.3. *If E is non-split then $\kappa^{(1)}_{a,b,c} = 0$ and $\kappa^{(1)}_{a,b,c}(q) = 0$.*

Lemma B.4.   • *If $E = F \times K$ then*

$$\int\limits_{(\varpi^{-1}\mathcal{O}^\times)^2} \psi\left(D_{(a,b,c)}r_2 + \frac{r_3^2}{r_2}\right) dr_2 \, dr_3 = 1 - q.$$

• *If E is a field then*

$$\int\limits_{(\varpi^{-1}\mathcal{O}^\times)^2} \psi\left(N_{(a,b,c)}\frac{r_2^2}{r_3} + D_{(a,b,c)}r_2 + \frac{r_3^2}{r_2}\right) dr_2 \, dr_3 = 1 - q.$$

• $\displaystyle \int\limits_{|x|,|y|,|xy| \le q} \psi(x+y) \, dx \, dy = -1.$

Remark B.1. The difference between the different types of étale cubic algebras boils down to the last two lemmas as the results for E split would be different.

Lemma B.5. *For $g \in G(F)$ assume that $E_k(g) = 0$ for any $k \ne n, n+1$. Then*

$$D_s^{\Psi_E}(g) = \frac{q^{-\left(s+\frac{7}{2}\right)n}}{\zeta_F\left(s+\frac{7}{2}\right)}\left(E_n(g) + E_{n+1}(g)q^{-s-\frac{7}{2}}\right).$$

Let $g = h_\alpha(t_1)h_\beta(t_2)x_\alpha(d) = x_\alpha(p)h_\alpha(t_1)h_\beta(t_2)$, where $p = \frac{dt_1^2}{t_2}$. Denote $|t_1| = q^{-n}$, $|t_2| = q^{-m}$ and $|p| = q^l$, we have $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$1 \le q^{k+n}$$
$$1, |p| \le q^{k+m-n}$$
$$1, |r_1|, |r_2| \le q^{k+2n-m}$$
$$1, |p|, |r_3 - pr_2|, |r_2 - pr_1| \le q^k$$
$$1, |p|, |p^2|, |2pr_3 - p^2r_2 - r_4|, |2pr_2 - r_3 - p^2r_1| \le q^{k+m-2n}$$
$$1, |r_1|, |r_2|, |r_3|, |r_2r_3 + r_5|, |r_2^2 - r_1r_3| \le q^{k+n-m}$$
$$1, |p|, |pr_1 - r_2|, |pr_2 - r_3|, |pr_3 - r_4|, |r_2r_4 - r_3^2 - pr_2r_3 - pr_5| \le q^{k-n}$$
$$|r_1r_4 - 2r_2r_3 + pr_2^2 - pr_1r_3 - r_5| \le q^{k-n}.$$

B.1. **Toral elements.** In this section we consider the case of $g \in UTK$. Since $E_k^{\Psi_E} \in \mathcal{M}_{\Psi_E}$ it is sufficient to consider $g = h_\alpha(t_1)h_\beta(t_2)$. Let $|t_1| = q^{-n}$ and $|t_2| = q^{-m}$.

Remark B.2. From Lemma 2.1 we have $D_s^{\Psi_E}(g) = 0$ unless

$$\text{(B.2)} \qquad \begin{cases} t_1, t_2, \frac{t_2}{t_1}, \frac{t_1^3}{t_2} \in \mathcal{O}, & N_{(a,b,c)} = 0 \\ t_1, t_2, \frac{t_2}{t_1}, \frac{t_1^3}{t_2}, \frac{t_2^2}{t_1^3} \in \mathcal{O}, & N_{(a,b,c)} \in \mathcal{O}^\times \end{cases}.$$

Thus we may assume that $|t_2| \leq |t_1| \leq 1$.

Furthermore, $U_k(g) = \emptyset$ unless $k \geq n, m - n$ and we have $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |r_2 r_3 + r_5| \leq q^{k+n-m}$$

$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - 2r_2 r_3 - r_5| \leq q^{k-n}.$$

Remark B.3. As an analogue of [12, Lemma B.2], note that

$$\int_{U_k(g)} \Psi_E(u)\, du = \int_{\widehat{U_k(g)}} \Psi_E(u)\, du,$$

where

$$\widehat{U_k(g)} = \left\{ u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) : \left|r_4 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1\right| \leq q \right\}.$$

We now split the computation to two cases, $N_{(a,b,c)} = 0$ or $N_{(a,b,c)} \in \mathcal{O}^\times$.

B.1.1. $N_{(a,b,c)} = 0$.
Case of $m < 2n$: We write

$$z := r_1 r_4 - 2r_2 r_3 - r_5.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, \left|r_1 r_3 - r_2^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k+n-m}$$

$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q^{k-n}.$$

- $\underline{k = n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1| \leq q^{2n-m}$$

$$|r_2|, |r_3|, |r_4|, |z| \leq 1.$$

Hence

$$E_n^{\Psi_E}(g) = \int_{\varpi^{m-2n}\mathcal{O}} dr_1 \int_{\mathcal{O}^4} \psi\left(r_4 + D_{(a,b,c)r_2}\right) dr_2\, dr_3\, dr_4\, dz = q^{2n-m}.$$

- $\underline{k = n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_1 r_3|, |r_1 r_4| \leq q^{2n-m+1}$$

$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q.$$

Note that since $|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q$, if $|r_3| = q$ then also $|r_2| = |r_4| = q$.
  - $\underline{|r_3|, |r_4| \leq 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1| \leq q^{2n-m+1}$$

$$|r_2|, |z| \leq q.$$

– $|r_3| \leq 1, |r_4| = q$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1| \leq q^{2n-m}$$
$$|r_2| \leq 1$$
$$|z| \leq q.$$

– $|r_3| = q$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1| \leq q^{2n-m}$$
$$|r_2|, |r_2 r_4 - r_3^2|, |z| \leq q.$$

We make a change of variables

$$x = r_2 r_4 - r_3^2$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1| \leq q^{2n-m}$$
$$|x|, |z| \leq q.$$

In conclusion,

$$E_{n+1}^{\Psi_E}(g) = q^{2n-m+2} \int_{\mathcal{O}} \psi(r_4)\, dr_4 \int_{\varpi^{-1}\mathcal{O}} \psi\left(D_{(a,b,c)} r_2\right) dr_2 +$$

$$q^{2n-m+1} \int_{\mathcal{O}} \psi\left(D_{(a,b,c)} r_2\right) dr_2 \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_4)} dr_4 +$$

$$q^{2n-m+1} \int_{(\varpi^{-1}\mathcal{O}^\times)^2} \psi\left(D_{(a,b,c)} r_2 + \frac{r_3^2}{r_2}\right) dr_2\, dr_3 = -q^{2n-m+2}.$$

- $k > n+1$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_1 r_3 - r_2^2|, |r_1 r_4 - r_2 r_3| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, |r_2 r_4 - r_3^2|, |z| \leq q^{k-n}.$$

We make a change of variables

$$x = r_4 + D_{(a,b,c)} r_2$$

So that $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_1 r_3 - r_2^2|, |r_1 \left(x - D_{(a,b,c)} r_2\right) - r_2 r_3| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_2 \left(x - D_{(a,b,c)} r_2\right) - r_3^2|, |z| \leq q^{k-n}.$$

– $|x|, |r_2| \leq 1$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_1 r_3| \leq q^{k+n-m}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $|x| \leq 1, 1 < |r_2| \leq q^{\frac{k-n}{2}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1 r_2|, |r_1 r_3| \leq q^{k+n-m}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $|x| \leq 1, q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_1 r_3 - r_2^2|, |D_{(a,b,c)} r_1 r_2 + r_2 r_3| \leq q^{k+n-m}$$
$$|r_3|, |D_{(a,b,c)} r_2^2 + r_3^2|, |z| \leq q^{k-n}.$$

We make a change of variables

$$y = D_{(a,b,c)} r_1 r_2 + r_2 r_3$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|y| \leq q^{k+n-m}$$
$$|r_3| = |r_2|$$
$$|D_{(a,b,c)} r_2^2 + r_3^2|, |z| \leq q^{k-n}.$$

- $|x| = q, |x - D_{(a,b,c)} r_2| \leq 1$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_1 r_3| \leq q^{k+n-m}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $|x| = q, |x - D_{(a,b,c)} r_2| = q$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \leq q^{k+n-m-1}$$
$$|r_1 r_3| \leq q^{k+n-m}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $|x| = q, q < |r_2| \leq q^{\frac{k-n}{2}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1 r_2|, |r_1 r_3| \leq q^{k+n-m}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $|x| = q, q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n-1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_1 r_3 - r_2^2|, |r_1(x - D_{(a,b,c)} r_2) - r_2 r_3| \leq q^{k+n-m}$$
$$|r_3| = |r_2|$$
$$|D_{(a,b,c)} r_2^2 + r_3^2|, |z| \leq q^{k-n}.$$

We make a change of variables

$$y = r_1(x - D_{(a,b,c)} r_2) - r_2 r_3$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|y| \leq q^{k+n-m}$$
$$|r_3| = |r_2|$$
$$\left|D_{(a,b,c)}r_2^2 + r_3^2\right|, |z| \leq q^{k-n}.$$

$-$ $\underline{|x| = q, |r_2| = q^{k-n}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, \left|r_1 r_3 - r_2^2\right|, \left|r_1\left(x - D_{(a,b,c)}r_2\right) - r_2 r_3\right| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, \left|r_2\left(x - D_{(a,b,c)}r_2\right) - r_3^2\right|, |z| \leq q^{k-n}.$$

We make a change of variables

$$y = r_1\left(x - D_{(a,b,c)}r_2\right) - r_2 r_3$$
$$r_2' = r_2 - \frac{x}{2D_{(a,b,c)}}.$$

Note that $|r_3| = |r_2'|$. It holds that $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|y| \leq q^{k+n-m}$$
$$|r_3| = |r_2'|$$
$$\left|D_{(a,b,c)}\left(r_2'\right)^2 + r_3^2\right|, |z| \leq q^{k-n}.$$

Denote $l = \lfloor \frac{k-n}{2} \rfloor$, it holds that

$$E_k^{\Psi_E}(g) = q^{k-n}\left(\kappa(k+n-m, l, k+n-m) + \sum_{j=1}^{l} q^j\left(1-q^{-1}\right)\kappa(k+n-m-j, l, k+n-m)\right.$$

$$+ q^{k+n-m}\sum_{j=l+1}^{k-n}\kappa_{(a,b,c)}^{(k-n)}\left(q^j\right)\right) - \left(\kappa(k+n-m, l, k+n-m) + \right.$$

$$\left.\sum_{j=1}^{l} q^j\left(1-q^{-1}\right)\kappa(k+n-m-j, l, k+n-m) + q^{k+n-m}\sum_{j=l+1}^{k-n}\kappa_{(a,b,c)}^{(k+n-m)}\left(q^j\right)\right) = 0.$$

Case of $m = 2n$: We write

$$z := r_2 r_3 + r_5$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k-n}.$$

We compute $E_k^{\Psi_E}(g)$ for different values of k.

- $\underline{k = n}$: In this case , $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_4|, |x|, |z| \leq 1.$$

and hence

$$E_n^{\Psi_E}(g) = \int_{\mathcal{O}^5} \psi\left(r_4 + D_{(a,b,c)}r_2\right) dr_1\ dr_2\ dr_3\ dr_4\ dz = 1.$$

- $\underline{k = n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, |r_1 r_3 - r_2^2|, |r_2 r_4 - r_3^2|, |r_1 r_4 - r_2 r_3| \le q.$$

  Note that from $|r_1 r_3 - r_2^2|, |r_2 r_4 - r_3^2| \le q$ it follows that $|r_2| = q$ if and only if $|r_3| = q$, in which case $|r_1| = |r_4| = q$ also.

  - $\underline{|r_2|, |r_3| \le 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_4|, |z|, |r_1 r_4| \le q.$$

  $\underline{|r_2|, |r_3| = q}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_4|, |z|, |r_1 r_3 - r_2^2|, |r_2 r_4 - r_3^2|, |r_1 r_4 - r_2 r_3| \le q.$$

  We make change of variables

$$x = r_1 r_3 - r_2^2$$
$$y = r_4 r_2 - r_3^2.$$

  and hence $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|x|, |y|, |z| \le q$$

  The other inequalities are satisfied immediately.

  In conclusion, we have

$$E_{n+1}^{\Psi_E}(g) = \int_{\varpi^{-1}\mathcal{O}} dz \int_{\mathcal{O}^2} dr_2\ dr_3 \int_{|r_1|, |r_4|, |r_1 r_4| \le q} \psi(r_4)\, dr_1\ dr_4 +$$

$$+ \int_{\varpi^{-1}\mathcal{O}} dz \int_{\varpi^{-1}\mathcal{O}^\times} dr_3 \int_{\varpi^{-1}\mathcal{O}^\times} \psi(D_{(a,b,c)} r_2)\, dr_2 \int_{\varpi^{-1}\mathcal{O}} \frac{dx}{|r_3|} \int_{\varpi^{-1}\mathcal{O}} \psi\left(\frac{y + r_3^2}{r_2}\right) \frac{dy}{|r_2|} =$$

$$= q(q-1) + \int_{(\varpi^{-1}\mathcal{O}^\times)^2} \psi\left(D_{(a,b,c)} r_2 + \frac{r_3^2}{r_2}\right) dr_2\ dr_3 \int_{\varpi^{-1}\mathcal{O}} \psi\left(\frac{y}{r_2}\right) dy = 0.$$

- $\underline{k > n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, |r_1 r_3 - r_2^2|, |r_2 r_4 - r_3^2|, |r_1 r_4 - r_2 r_3| \le q^{k-n}.$$

  We make a change of variables

$$x = r_4 + D_{(a,b,c)} r_2.$$

  We then have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$|x| \le q$

$|r_1|, |r_2|, |r_3|, |z|, |r_1 r_3 - r_2^2|, |r_2(x - D_{(a,b,c)} r_2) - r_3^2|, |r_1(x - D_{(a,b,c)} r_2) - r_2 r_3| \le q^{k-n}.$

  - $\underline{|x|, |r_2| \le 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_3| \le q^{\frac{k-n}{2}}$$
$$|r_1|, |r_1 r_3|, |z| \le q^{k-n}.$$

  - $\underline{|x| \le 1, 1 < |r_2| \le q^{\frac{k-n}{2}}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_3| \le q^{\frac{k-n}{2}}$$
$$|z|, |r_1 r_2|, |r_1 r_3| \le q^{k-n}.$$

- $|x| \leq 1, q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n}$: In this case, $u\,(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k\,(g)}$ if and only if

$$|r_1|, |r_3|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right|, \left|\left(D_{(a,b,c)} r_1 + r_3\right) r_2\right| \leq q^{k-n}.$$

Note that from $\left|D_{(a,b,c)} r_2^2 + r_3^2\right| \leq q^{k-n}$ it follows that $|r_2| = |r_3|$. We make a change of variables

$$y = \left(D_{(a,b,c)} r_1 + r_3\right) r_2$$

to have $u\,(r_1, r_2, r_3, r_4, r_5) \in U_k\,(g)$ if and only if

$$|r_2| = |r_3|$$
$$|y|, |z|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right| \leq q^{k-n}.$$

- $|x| = q, \left|x - D_{(a,b,c)} r_2\right| \leq 1$: In this case $|r_2| \leq q \leq q^{k-n}$. We have $u\,(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k\,(g)}$ if and only if

$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|r_1|, |z|, |r_1 r_3| \leq q^{k-n}.$$

- $|x| = q, \left|x - D_{(a,b,c)} r_2\right| = q$: In this case $|r_2| \leq q \leq q^{k-n}$. We have $u\,(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k\,(g)}$ if and only if

$$|r_1| \leq q^{k-n-1}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z|, |r_1 r_3| \leq q^{k-n}.$$

- $|x| = q, q < |r_2| \leq q^{\frac{k-n}{2}} \leq q^{k-n-1}$: Note that in this case (if it happens) $\left|x - D_{(a,b,c)} r_2\right| = |r_2|$ and hence $u\,(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k\,(g)}$ if and only if

$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z|, |r_1 r_3|, |r_1 r_2| \leq q^{k-n}.$$

- $|x| = q, q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n-1}$: In this case, $u\,(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k\,(g)}$ if and only if

$$|r_1|, |r_3|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right|, \left|r_1\left(x - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \leq q^{k-n}.$$

Note that here $|r_1| = |r_2| = |r_3|$ since $\left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right| \leq q^{k-n}$ and $q^{\frac{k-n}{2}} < |r_2|$. We make a change of variables

$$y = r_1\left(x - D_{(a,b,c)} r_2\right) - r_2 r_3$$

and then $u\,(r_1, r_2, r_3, r_4, r_5) \in U_k\,(g)$ if and only if

$$|r_2| = |r_3|$$
$$|y|, |z|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right| \leq q^{k-n}.$$

- $|x| = q, |r_2| = q^{k-n}$: In this case, $u\,(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k\,(g)}$ if and only if

$$|r_1|, |r_3|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2\left(x - D_{(a,b,c)} r_2\right) - r_3^2\right|, \left|r_1\left(x - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \leq q^{k-n}.$$

Note that here $|r_1| = |r_2| = |r_3| = q^{k-n}$ since $\left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right| \leq q^{k-n}$ and $q^{\frac{k-n}{2}} < |r_2|$. We make a change of variables

$$y = r_1 \left(x - D_{(a,b,c)} r_2\right) - r_2 r_3$$
$$r_2' = r_2 - \frac{x}{2 D_{(a,b,c)}}.$$

Note that $|r_2| = |r_2'|$. We then have $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k(g)}$ if and only if

$$|y|, |r_3|, |z|, \left|D_{(a,b,c)} \left(r_2'\right)^2 + r_3^2\right| \leq q^{k-n}.$$

Denote $l = \lfloor \frac{k-n}{2} \rfloor$, it holds that

$$E_k^{\Psi_E}(g) = q^{k-n} \left(\kappa(k-n, l, k-n) + \sum_{j=1}^{l} q^j \left(1 - q^{-1}\right) \kappa(k-n-j, l, k-n) \right.$$

$$+ q^{k-n} \sum_{j=l+1}^{k-n} \kappa_{(a,b,c)}^{(k-n)}\left(q^j\right) \bigg) - \bigg(\kappa(k-n, l, k-n) + q\left(1 - q^{-1}\right) \kappa(k-n-1, l, k-n) +$$

$$\sum_{j=2}^{l} q^j \left(1 - q^{-1}\right) \kappa(k-n-j, l, k-n) + q^{k-n} \sum_{j=l+1}^{k-n} \kappa_{(a,b,c)}^{(k-n)}\left(q^j\right) \bigg) = 0.$$

<u>Case of $m > 2n$</u>: Since the proof of this case is identical to the case of $N_{(a,b,c)} \in \mathcal{O}^\times$, we write here

$$\Psi_E\left(u\left(r_1, r_2, r_3, r_4, r_5\right)\right) = \psi\left(r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1\right)$$

so that the proof will fit both cases. We write

$$z := r_2 r_3 + r_5$$

In this case, $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |z| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, \left|r_1 r_4 - r_2 r_3\right| \leq q^{k-n}.$$

- <u>$k = m - n$</u>: In this case, $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, |z| \leq 1$$
$$|r_4| \leq q^{m-2n}.$$

and since $m - 2n \geq 1$ it holds that

$$E_n^{\Psi_E}(g) = \int_{|r_4| \leq q^{m-2n}} \psi(r_4) \, dr_4 = 0.$$

- <u>$k = m - n + 1$</u>: In this case, $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |z| \leq q$$
$$|r_4|, |r_2 r_4|, |r_1 r_4| \leq q^{m-2n+1}.$$

  - <u>$|r_1|, |r_2| \leq 1$</u>: In this case, $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k(g)$ if and only if

$$|r_3|, |z| \leq q$$
$$|r_4| \leq q^{m-2n+1}.$$

- $|r_2| \leq 1, |r_1| = q$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_3| \leq 1$$
$$|z| \leq q$$
$$|r_4| \leq q^{m-2n}.$$

- $|r_2| = q$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1| = |r_3| = q$$
$$|z| \leq q$$
$$|r_4| \leq q^{m-2n}.$$

In conclusion, we have

$$E_{n+1}^{\Psi_E}(g) = q^2 \int_{|r_4| \leq q^{m-2n+1}} \psi(r_4)\, dr_4 + q(q-1) \int_{|r_4| \leq q^{m-2n}} \psi(r_4)\, dr_4 +$$

$$q(q-1) \int_{|r_1|=|r_2|=q} \psi\left(D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1\right) dr_1\, dr_2\, dr_4 \int_{|r_4| \leq q^{m-2n}} \psi(r_4)\, dr_4 = 0.$$

- $\underline{k > m - n + 1}$: In this case , $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |z| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k-n}.$$

We make a change of variables

$$x = r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |z| \leq q^{k+n-m}$$
$$\left|r_2\left(N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right) - r_3^2\right|, \left|r_1\left(N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \leq q^{k-n}.$$

Denote by $C \subset F^3$ the set of elements $(r_1, r_2, r_3) \in C$ such that

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right| \leq q^{k+n-m}$$
$$\left|r_2\left(N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right) - r_3^2\right|, \left|r_1\left(N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \leq q^{k-n}.$$

It then holds that

$$E_k^{\Psi_E}(g) = q^{k+n-m} \lambda(C) \int_{|r_4| \leq q} \psi(r_4)\, dr_4 = 0.$$

B.1.2. $N_{(a,b,c)} \in \mathcal{O}^\times$.
Case of $m < 2n$: We write

$$z := r_1 r_4 - 2 r_2 r_3 - r_5.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, \left|r_1 r_3 - r_2^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q^{k-n}.$$

- $\underline{k = n:}$ In this case , $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k\left(g\right)$ if and only if

$$|r_1| \leq q^{2n-m}$$
$$|r_2|, |r_3|, |r_4|, |z| \leq 1$$

and hence

$$E_n^{\Psi_E}\left(g\right) = \int\limits_{|r_1| \leq q^{2n-m}} \psi\left(N_{(a,b,c)}r_1\right) dr_1 = 0,$$

since $2n - m \geq 1$.

- $\underline{k = n+1:}$ In this case , $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k\left(g\right)$ if and only if

$$|r_1|, |r_1 r_3|, |r_1 r_4| \leq q^{2n-m+1}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q.$$

  - $\underline{|r_4| \leq 1:}$ In this case, $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k\left(g\right)$ if and only if

$$|r_1| \leq q^{2n-m+1}$$
$$|r_2|, |z| \leq q$$
$$|r_3| \leq 1.$$

  - $\underline{|r_4| = q:}$ In this case, $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k\left(g\right)$ if and only if

$$|r_1| \leq q^{2n-m}$$
$$|r_2|, |r_3|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q.$$

In conclusion

$$E_{n+1}^{\Psi_E}\left(g\right) = q^2 \int\limits_{|r_1| \leq q^{2n-m+1}} \psi\left(N_{(a,b,c)}r_1\right) dr_1 +$$

$$q \int\limits_{|r_1| \leq q^{2n-m}} \psi\left(N_{(a,b,c)}r_1\right) dr_1 \int\limits_{\substack{|r_4|=q \\ |r_2|,|r_3|,\left|r_2 r_4 - r_3^2\right|,|z| \leq q}} \psi\left(r_4 + D_{(a,b,c)}r_2\right) dr_2 \, dr_3 \, dr_4 = 0.$$

- $\underline{k > n+1:}$ In this case , $u\left(r_1, r_2, r_3, r_4, r_5\right) \in U_k\left(g\right)$ if and only if

$$|r_1|, \left|r_1 r_3 - r_2^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q^{k-n}.$$

We make a change of variables

$$x = r_4 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1$$

and then $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$|x| \leq q$

$\left|\left(r_4 + D_{(a,b,c)}r_2\right) r_3 - N_{(a,b,c)}r_2^2\right|, \left|\left(r_4 + D_{(a,b,c)}r_2\right) r_4 - N_{(a,b,c)r_2 r_3}\right| l \leq q^{k+n-m}$

$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q^{k-n}.$

We denote by $C \subset F^3$ the set of all elements $\left(r_2, r_3, r_4\right)$ such that

$\left|\left(r_4 + D_{(a,b,c)}r_2\right) r_3 - N_{(a,b,c)}r_2^2\right|, \left|\left(r_4 + D_{(a,b,c)}r_2\right) r_4 - N_{(a,b,c)r_2 r_3}\right| \leq q^{k+n-m}$

$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right| \leq q^{k-n}.$

It holds that

$$E_k^{\Psi_E}(g) = q^{k-n}\lambda(C)\int\limits_{|x|\leq q}\psi(x)\,dx = 0.$$

Case of $m = 2n$: We write

$$z := r_2 r_3 + r_5.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, |r_1 r_3 - r_2^2|, |r_2 r_4 - r_3^2|, |r_1 r_4 - r_2 r_3| \leq q^{k-n}.$$

- $\underline{k = n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z| \leq 1$$

  and hence

$$E_n^{\Psi_E}(g) = 1.$$

- $\underline{k = n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, |r_1 r_3 - r_2^2|, |r_2 r_4 - r_3^2|, |r_1 r_4 - r_2 r_3| \leq q.$$

  Note that $|r_2| = q$ if and only if $|r_3| = q$, in which case $|r_1| = |r_4| = q$.
  - $\underline{|r_2|, |r_3| \leq 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_4|, |z|, |r_1 r_4| \leq q.$$

  $\underline{|r_2| = |r_3| = q}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_4|, |z|, |r_1 r_3 - r_2^2|, |r_2 r_4 - r_3^2|, |r_1 r_4 - r_2 r_3| \leq q.$$

  We make a change of variables

$$x = r_1 r_3 - r_2^2$$
$$y = r_2 r_4 - r_3^2.$$

  It then holds that $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|x|, |y|, |z| \leq q.$$

  In conclusion

$$E_{n+1}^{\Psi_E}(g) = q\int\limits_{|r_1|, |r_4|, |r_1 r_4| \leq q}\psi\left(r_4 + D_{(a,b,c)}r_2\right)dr_2\,dr_4 +$$

$$q\int\limits_{\substack{|r_2|=|r_3|=q \\ |x|, |y| \leq q}}\psi\left(\frac{y + r_3^2}{r_2} + D_{(a,b,c)}r_2 + N_{(a,b,c)\frac{x+r_2^2}{r_3}}\right)dr_2\,dr_3\,\frac{dx}{q}\,\frac{dy}{q} = -q + q(1-q) = -q^2.$$

- $\underline{k > n+1}$: We make a change of variables

$$x = r_4 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1.$$

  In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_2|, |r_3|, |z|, \left|r_1\left(x - D_{(a,b,c)}r_2 + N_{(a,b,c)}r_1\right) - r_2 r_3\right| \leq q^{k-n}$$
$$\left|r_1 r_3 - r_2^2\right|, \left|r_2\left(x - D_{(a,b,c)}r_2 + N_{(a,b,c)}r_1\right) - r_3^2\right| \leq q^{k-n}.$$

- $|r_2| \leq q^{\frac{k-n}{2}}$: In this case , $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $q^{\frac{k-n}{2}} < |r_2| < q^{k-n}$: In this case , $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$$|x| \leq q$$
$$|r_1| \leq q^{k-n-1}$$
$$|r_3|, |z|, \left|r_1\left(N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_2 r_3\right| \leq q^{k-n}$$
$$\left|r_1 r_3 - r_2^2\right|, \left|r_2\left(N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_3^2\right| \leq q^{k-n}.$$

- $|r_2| = q^{k-n}$: In this case , $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_2|, |r_3|, |z|, \left|r_1\left(x - D_{(a,b,c)}r_2 + N_{(a,b,c)}r_1\right) - r_2 r_3\right| \leq q^{k-n}$$
$$\left|r_1 r_3 - r_2^2\right|, \left|r_2\left(x - D_{(a,b,c)}r_2 + N_{(a,b,c)}r_1\right) - r_3^2\right| \leq q^{k-n}.$$

First note that $|r_1| = |r_2| = |r_3| = q^{k-n}$, hence we may write $\epsilon = \frac{r_2}{r_1}$ with $|\epsilon| = 1$. Consider $\left|r_1\left(x - D_{(a,b,c)}r_2 + N_{(a,b,c)}r_1\right) - r_2 r_3\right| \leq q^{k-n}$, multiplying by ϵ^3 and dividing by r_2^2 we have

$$\left|\left(\epsilon^3 + D_{(a,b,c)}\epsilon - N_{(a,b,c)}\right) - \frac{\epsilon x}{r_2}\right| \leq \frac{1}{q^{k-n}} < 1.$$

Since E/F is unramified and $\epsilon \in \mathcal{O}$, it holds that $\left|\epsilon^3 + D_{(a,b,c)}\epsilon - N_{(a,b,c)}\right| = 1$ and hence also $\left|\frac{\epsilon x}{r_2}\right| = 1$ contradicting the fact that $\left|\frac{\epsilon x}{r_2}\right| \leq \frac{q}{q^{k-n}} < 1$.

Let $l = \lfloor \frac{k-n}{2} \rfloor$. Also, let $C \subset F^3$ be the set such that $(r_1, r_2, r_3) \in C$ if and only if

$$q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n-1}$$
$$|r_1| \leq q^{k-n-1}$$
$$|r_3|, \left|r_1\left(N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_2 r_3\right| \leq q^{k-n}$$
$$\left|r_1 r_3 - r_2^2\right|, \left|r_2\left(N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_3^2\right| \leq q^{k-n}.$$

It then holds that

$$E_k^{\Psi_E}\left(g\right) = q^{k-n}\left(q^{l+k-n} + \lambda\left(C\right)\right) \int_{|x| \leq q} \psi\left(x\right) dx = 0.$$

<u>Case of $m > 2n$</u>: This was already treated in the discussion of the case $N_{(a,b,c)} = 0$.

B.2. **Non-Toral Elements.** In this section we consider the case of $g \notin UTK$. Since $E_k^{\Psi_E} \in \mathcal{M}_{\Psi_E}$ it is sufficient to consider $g = h_\alpha(t_1)h_\beta(t_2)x_\alpha\left(d\right) = x_\alpha\left(p\right)h_\alpha(t_1)h_\beta(t_2)$, where $p = \frac{t_1^2 d}{t_2}$. Let $|t_1| = q^{-n}$, $|t_2| = q^{-m}$ and $|p| = q^l$.

Remark B.4. • Let $E = F \times K$ and $\eta = w_\alpha h_\beta(bc) \in G(\mathcal{O})$. Recall that according to **(CT)** we assume that $a = 0$ and $b, c \neq 0$. Since $E_k \in \mathcal{H}$, it holds that

$$E_k^{\Psi_{(0,b,c)}}(g) = \int\limits_{U(F)} E_k(ug)\left(\Psi_{(0,b,c)}(u)\right) du = \int\limits_{U(F)} E_k\left(\eta^{-1}ug\eta\right)\left(\Psi_{(0,b,c)}(u)\right) du =$$

$$= \int\limits_{U(F)} E_k\left(u'\eta^{-1}g\eta\right)\left(\Psi_{(0,b,c)}\left(\eta u'\eta^{-1}\right)\right) du' = \int\limits_{U(F)} E_k(u'g')\left(\Psi_{\left(0,\frac{1}{b},\frac{1}{c}\right)}\right)(u')\, du' = E_k^{\Psi_{\left(0,\frac{1}{b},\frac{1}{c}\right)}}(g'),$$

where $g' = x_\alpha(p')\, h_\alpha(\frac{bct_2}{dt_1})h_\beta(t_2)$ and $p' = bc\frac{t_2}{dt_1^2}$.

• Let E be a cubic field extension and $\eta = w_\alpha h_\beta(abc) \in G(\mathcal{O})$. Recall that according to **(CT)** we assume that $a, b, c \neq 0$. As above, it holds that

$$E_k^{\Psi_E}(g) = E_k^{\Psi_{\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)}}(g'),$$

where $g' = x_\alpha(p')\, h_\alpha(\frac{abct_2}{dt_1})h_\beta(t_2)$ and $p' = abc\frac{t_2}{dt_1^2}$.

In both cases $|p'| = \frac{1}{|p|}$ and hence we may assume from now on that $|p| \leq 1$ since (a, b, c) is arbitrary. Therefore we assume for the rest of this section that $|p| \leq 1$. Also note that under this assumption, it follows that $\left|\frac{t_1^2}{t_2}\right| < 1$.

Remark B.5. From Lemma 2.1, and under the assumption that $|p| \leq 1$, we have $D_s^{\Psi_E}(g) = 0$ unless

$$\frac{t_1^3}{t_2}, t_1, \frac{t_2}{t_1^3} \in \mathcal{O}.$$

Furthermore, $U_k(g) = \emptyset$ unless $k \geq n$. We have $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$k \geq n, m - n$$
$$|r_1|, |r_2|, |r_3|, |r_2 r_3 + r_5|, \left|r_2^2 - r_1 r_3\right| \leq q^{k+n-m}$$
$$|pr_1 - r_2|, |pr_2 - r_3|, |pr_3 - r_4|, \left|r_2 r_4 - r_3^2 - pr_2 r_3 - pr_5\right| \leq q^{k-n}$$
$$\left|r_1 r_4 - 2r_2 r_3 + pr_2^2 - pr_1 r_3 - r_5\right| \leq q^{k-n}.$$

Remark B.6. As an analogue of [12, Lemma B.2], note that

$$\int\limits_{U_k(g)} \Psi_E(u)\, du = \int\limits_{\widehat{U_k(g)}} \Psi_E(u)\, du,$$

where

$$\widehat{U_k(g)} = \left\{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) : \left|r_4 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1\right| \leq q\right\}.$$

We now split the computation to two cases, $N_{(a,b,c)} = 0$ or $N_{(a,b,c)} \in \mathcal{O}^\times$.

B.2.1. $N_{(a,b,c)} = 0$. We write

$$x = r_4 + D_{(a,b,c)}r_2$$
$$z = r_2 r_3 + r_5.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_2|, |r_3|, |z|, |r_2^2 - r_1 r_3| \leq q^{k+n-m}$$
$$|pr_1 - r_2|, |pr_2 - r_3|, |pr_3 - x + D_{(a,b,c)} r_2|, |r_2 x - D_{(a,b,c)} r_2^2 - r_3^2 - pz| \leq q^{k-n}$$
$$\left| r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + pr_2^2 - pr_1 r_3 - z \right| \leq q^{k-n}.$$

Let $\kappa = p^2 + D_{(a,b,c)}$.
Case of $l < 0$:

- $\underline{k = n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_2|, |r_3|, |z|, |r_2^2 - r_1 r_3| \leq q^{2n-m}$$
$$|pr_1 - r_2|, |pr_2 - r_3|, |pr_3 - x + D_{(a,b,c)} r_2|, |r_2 x - D_{(a,b,c)} r_2^2 - r_3^2 - pz| \leq 1$$
$$\left| r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + pr_2^2 - pr_1 r_3 - z \right| \leq 1.$$

  - $\underline{|x| \leq 1}$: We make a change of variables

    $$y = r_1 x - D_{(a,b,c)} r_1 r_2 - z$$

    and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

    $$|r_1| \leq q^{-l}$$
    $$|r_2|, |r_3|, |y| \leq 1.$$

  - $\underline{|x| = q}$: We make a change of variables

    $$y = r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + pr_2^2 - pr_1 r_3 - z$$
    $$l_1 = pr_1 - r_2$$
    $$l_2 = D_{(a,b,c)} r_2 - x$$

    and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

    $$|l_1|, |l_2|, |r_3|, |y| \leq 1.$$

  In conclusion

  $$E_n^{\Psi_E}(g) = q^{-l} \int_{|x| \leq 1} \psi(x)\, dx + q^{-l} \int_{|x| = q} \psi(x)\, dx = 0.$$

- $\underline{k > n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

  $$|x| \leq q$$
  $$|z|, |r_2^2 - r_1 r_3| \leq q^{k+n-m}$$
  $$|r_1|, |r_2|, |r_3|, |r_2 x - D_{(a,b,c)} r_2^2 - r_3^2 - pz| \leq q^{k-n}$$
  $$\left| r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + pr_2^2 - pr_1 r_3 - z \right| \leq q^{k-n}.$$

We make a change of variables

$$y = r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + p r_2^2 - p r_1 r_3 - z$$
$$l_1 = r_2 - p r_1$$
$$l_2 = p r_2 - r_3$$
$$l_3 = D_{(a,b,c)} r_2 + p r_3 - x$$

and then $u\,(r_1, r_2, r_3, r_4, r_5) \in U_k\,(g)$ if and only if

$$|x| \le q$$
$$|p\,(p l_2 + l_3) \cdot l_2 + (\kappa l_1 + p l_2 + l_3)\, l_3| \le q^{k+n-m+l}$$
$$\left| p\,(p l_2 + l_3)^2 - (\kappa l_1 + p l_2 + l_3)\,\left(p l_3 - D_{(a,b,c)} l_2\right) \right| \le q^{k+n-m+l}$$
$$|l_1|\,, |p l_2 + l_3|\,, \left|p l_3 - D_{(a,b,c)} l_2\right|\,, |y|\,, \left|l_1 l_3 + l_2^2\right| \le q^{k-n}.$$

Let $C \subseteq F^3$ be the subset such that $(l_1, l_2, l_3) \in C$ if and only if

$$|p\,(p l_2 + l_3) \cdot l_2 + (\kappa l_1 + p l_2 + l_3)\, l_3| \le q^{k+n-m+l}$$
$$\left| p\,(p l_2 + l_3)^2 - (\kappa l_1 + p l_2 + l_3)\,\left(p l_3 - D_{(a,b,c)} l_2\right) \right| \le q^{k+n-m+l}$$
$$|l_1|\,, |p l_2 + l_3|\,, \left|p l_3 - D_{(a,b,c)} l_2\right|\,, \left|l_1 l_3 + l_2^2\right| \le q^{k-n}.$$

Hence

$$E_k^{\Psi_E}\,(g) = q^{k-n} \lambda\,(C) \int\limits_{|x| \le q} \psi\,(x)\, dx = 0.$$

Case of $l = 0$:

- $\underline{k = n}$: We make a change of variables

$$y = r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + p r_2^2 - p r_1 r_3 - z$$
$$l_1 = p r_1 - r_2$$
$$l_2 = \kappa r_2 - x$$
$$l_3 = r_3 - p r_2.$$

In this case , $u\,(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k\,(g)}$ if and only if

$$|x| \le q$$
$$|y|\,, |l_1|\,, |l_2|\,, |l_3| \le 1$$

and hence

$$E_n\,(g) = \int\limits_{|x| \le q} \psi\,(x)\, dx = 0.$$

- $\underline{k > n}$: We make a change of variables

$$y = r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + p r_2^2 - p r_1 r_3 - z$$
$$l_1 = r_2 - p r_1$$
$$l_2 = p r_2 - r_3$$
$$l_3 = D_{(a,b,c)} r_2 + p r_3 - x.$$

In this case , $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$$|x| \leq q$$

$$\left|\frac{pl_2 + l_3}{\kappa} \cdot l_2 + \left(\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p}\right) \cdot l_3\right| \leq q^{k+n-m}$$

$$\left|\left(\frac{pl_2 + l_3}{\kappa}\right)^2 - \left(\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p}\right) \cdot \frac{pl_3 - D_{(a,b,c)}l_2}{\kappa}\right| \leq q^{k+n-m}$$

$$\left|\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p}\right|, \left|\frac{pl_2 + l_3}{\kappa}\right|, \left|\frac{pl_3 - D_{(a,b,c)}l_2}{\kappa}\right|, \left|l_1 l_3 + l_2^2\right|, |y| \leq q^{k-n}.$$

Let $C \subseteq F^3$ be the subset such that $(l_1, l_2, l_3) \in C$ if and only if

$$\left|\frac{pl_2 + l_3}{\kappa} \cdot l_2 + \left(\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p}\right) \cdot l_3\right| \leq q^{k+n-m}$$

$$\left|\left(\frac{pl_2 + l_3}{\kappa}\right)^2 - \left(\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p}\right) \cdot \frac{pl_3 - D_{(a,b,c)}l_2}{\kappa}\right| \leq q^{k+n-m}$$

$$\left|\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p}\right|, \left|\frac{pl_2 + l_3}{\kappa}\right|, \left|\frac{pl_3 - D_{(a,b,c)}l_2}{\kappa}\right|, \left|l_1 l_3 + l_2^2\right| \leq q^{k-n}.$$

Hence

$$E_k^{\Psi_E}(g) = q^{k-n}\lambda(C) \int_{|x| \leq q} \psi(x)\, dx = 0.$$

B.2.2. $N_{(a,b,c)} \in \mathcal{O}^\times$. We write

$$x = r_4 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1$$
$$z = r_1 r_4 - 2r_2 r_3 - r_5.$$

For any $k \geq n$, $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$$|x| \leq q$$

$$|r_1|, |r_2|, |r_3|, |z|, \left|r_2^2 - r_1 r_3\right| \leq q^{k+n-m}$$

$$|pr_1 - r_2|, |pr_2 - r_3|, \left|pr_3 - \left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right)\right|, \left|r_2\left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_3^2 - pz\right| \leq q^{k-n}$$

$$\left|r_1\left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_2 r_3 + pr_2^2 - pr_1 r_3 - z\right| \leq q^{k-n}.$$

Let $\kappa = p^3 + D_{(a,b,c)}p - N_{(a,b,c)} \in \mathcal{O}^\times$.

- $\underline{k = n}$: We note that

$$\kappa r_1 - x = p^2\left(pr_1 - r_2\right) + p\left(pr_2 - r_3\right) + \left(pr_3 - \left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right)\right)$$

Which means that $|\kappa r_1 - x| \leq 1$ and hence $|r_1|, |r_2|, |r_3| \leq q$. We make a change of variables

$$y = r_1\left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2 - pr_3\right) - r_2\left(pr_2 - r_3\right) - z$$
$$l_1 = \kappa r_1 - x$$
$$l_2 = r_2 - pr_1$$
$$l_3 = r_3 - pr_2.$$

One checks that $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$$|x| \leq q$$
$$\left|\kappa^2 l_2^2 + p\kappa l_1 l_2 - \kappa l_1 l_3\right| \leq q^{2n-m}$$
$$|l_1|, |l_2|, |l_3|, |y| \leq 1.$$

Let $C \subseteq F^3$ be the subset such that $(l_1, l_2, l_3) \in C$ if and only if

$$\left|\kappa^2 l_2^2 + p\kappa l_1 l_2 - \kappa l_1 l_3\right| \leq q^{2n-m}$$
$$|l_1|, |l_2|, |l_3| \leq 1.$$

Hence

$$E_n^{\Psi_E}\left(g\right) = \lambda\left(C\right) \int\limits_{|x| \leq q} \psi\left(x\right) dx = 0.$$

- $\underline{k > n}$: We note that

$$\kappa r_1 - x = p^2\left(pr_1 - r_2\right) + p\left(pr_2 - r_3\right) + \left(pr_3 - \left(x + N_E r_1 - D_E r_2\right)\right)$$

Which means that $|\kappa r_1 - x| \leq q^{k-n}$ and hence $|r_1|, |r_2|, |r_3| \leq q^{k-n}$. Moreover $\left|br_3 - \left(x + N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right)\right|$ q^{k-n} if and only if $|\kappa r_1 - x| \leq q^{k-n}$.

We make a change of variables

$$y = r_1\left(x - N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2 - pr_3\right) - r_2\left(pr_2 - r_3\right) - z$$
$$l_1 = \kappa r_1 - x$$
$$l_2 = r_2 - pr_1$$
$$l_3 = r_3 - pr_2.$$

One checks that $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$$|x| \leq q$$
$$\left|\left(\kappa l_2 + pl_1\right)^2 - l_1\left(\kappa l_3 + p\kappa l_2 + p^2 l_1\right)\right|, \left|l_1\left(l_1 + \left(p^2 + D_{(a,b,c)}\right) l_2 + pl_3\right) + \left(\kappa l_2 + pl_1\right) l_3\right| \leq q^{k+n-m}$$
$$|l_1|, |l_2|, |l_3|, |y|, \left|l_1\left(l_1 + \left(p^2 + D_{(a,b,c)}\right) l_2 + pl_3\right) - l_3^2\right| \leq q^{k-n}.$$

Let $C \subseteq F^3$ be the subset such that $(l_1, l_2, l_3) \in C$ if and only if

$$\left|l_1\left(l_1 + \left(p^2 + D_{(a,b,c)}\right) l_2 + pl_3\right) + \left(\kappa l_2 + pl_1\right) l_3\right| \leq q^{k+n-m}$$
$$|l_1|, |l_2|, |l_3|, \left|l_1\left(l_1 + \left(p^2 + D_{(a,b,c)}\right) l_2 + pl_3\right) - l_3^2\right| \leq q^{k-n}.$$

Hence

$$E_k^{\Psi_E}\left(g\right) = q^{k-n}\lambda\left(C\right) \int\limits_{|x| \leq q} \psi\left(x\right) dx = 0.$$

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School of Mathematics, Ben Gurion University of the Negev, POB 653, Be'er Sheva 84105, Israel

E-mail address: `avners@math.bgu.ac.il`